4.1/4.3

Suppose a group G acts on a set A.
Recall, given $a \in A$ we have

Stabilizer of a $\longrightarrow$ $G_a = \{ g \in G \mid g \cdot a = a \}$

The orbit of a is

$$G \cdot a = \{ g \cdot a \mid g \in G \}$$

Given a group G, we can have G act on $A = G$ by conjugation:

Given $g \in G$ and $a \in G$ we define

$$g \cdot a = gag^{-1}$$

By HW this is a group action

Two elements $a, b \in G$ are said to be <u>conjugate</u> if there exists $g \in G$ with $g \cdot a = b$, that is, $gag^{-1} = b$.

<u>Note:</u> If $\underbrace{gag^{-1} = b}_{g \cdot a = b}$, then $\underbrace{a = g^{-1}bg = g^{-1}b(g^{-1})^{-1} = g^{-1} \cdot b}_{g^{-1} \cdot b = a}$

If a and b are conjugate then $G \cdot a = G \cdot b$

<u>pf:</u> Suppose a and b are conjugate. Then there exists $g \in G$ with $g \cdot a = b$.

Let $x \in G \cdot a$.

So, $x = g_1 \cdot a$ for some $g_1 \in G$.

Since $g \cdot a = b$ we have $g^{-1} \cdot (g \cdot a) = g^{-1} \cdot b$.

So, $(g^{-1}g) \cdot a = g^{-1} \cdot b$.

So, $1 \cdot a = g^{-1} \cdot b$.

Thus, $a = g^{-1} \cdot b$.

So, $x = g_1 \cdot a = g_1 \cdot (g^{-1} \cdot b) = (g_1 g^{-1}) \cdot b \in G \cdot b$

So, $G \cdot a \subseteq G \cdot b$

Let $r \in G \cdot b$.

So, $r = g_2 \cdot b$ for some $g_2 \in G$.

Then, $r = g_2 \cdot (g \cdot a) = (g_2 g) \cdot a$

So, $r \in G \cdot a$.

Thus, $G \cdot b \subseteq G \cdot a$. $\blacksquare$

<u>Fact:</u> $a \in G \cdot a$

<u>pf:</u> $a = 1 \cdot a$. $\blacksquare$

<u>Prop:</u> If $G \cdot a = G \cdot b$, then a and b are conjugate.

<u>pf:</u> We know $b \in G \cdot b$.

So, if $G \cdot a = G \cdot b$, then $b \in G \cdot a$.

So, $b = g \cdot a$ for some $g \in G$.

Thus, a and b are conjugate. $\blacksquare$

<u>Summary:</u> a and b are conjugate iff $G \cdot a = G \cdot b$.

Gi

act

Giv

Two

if

Note:

When G acts on G by conjugation, $G \cdot a$ is called the <u>conjugacy class of a.</u>

<u>Ex:</u> $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$D_8 \cdot 1 = \{g \cdot 1 \mid g \in D_8\} = \{g 1 g^{-1} \mid g \in D_8\} = \{1\}$

$\cdot r = \{1r1^{-1}, rrr^{-1}, (r^2)r(r^2)^{-1}, (r^3)r(r^3)^{-1}, srs^{-1}, (sr)r(sr)^{-1}, (sr^2)r(sr^2)^{-1}, (sr^3)r(sr^3)^{-1}\}$

$= \{r, r, r, r, r^3, r^3, r^3, r^3\} = \{r, r^3\} = D_8 \cdot r^3$

$srs^{-1} = srs = ssr^{-1} = 1r^3 = r^3$

$sr^2 r r^{-2} s^{-1} = srs = r^3$

$srrr^{-1}s^{-1} = srs = r^3$

$D_8 \cdot 1 = \{1\}$

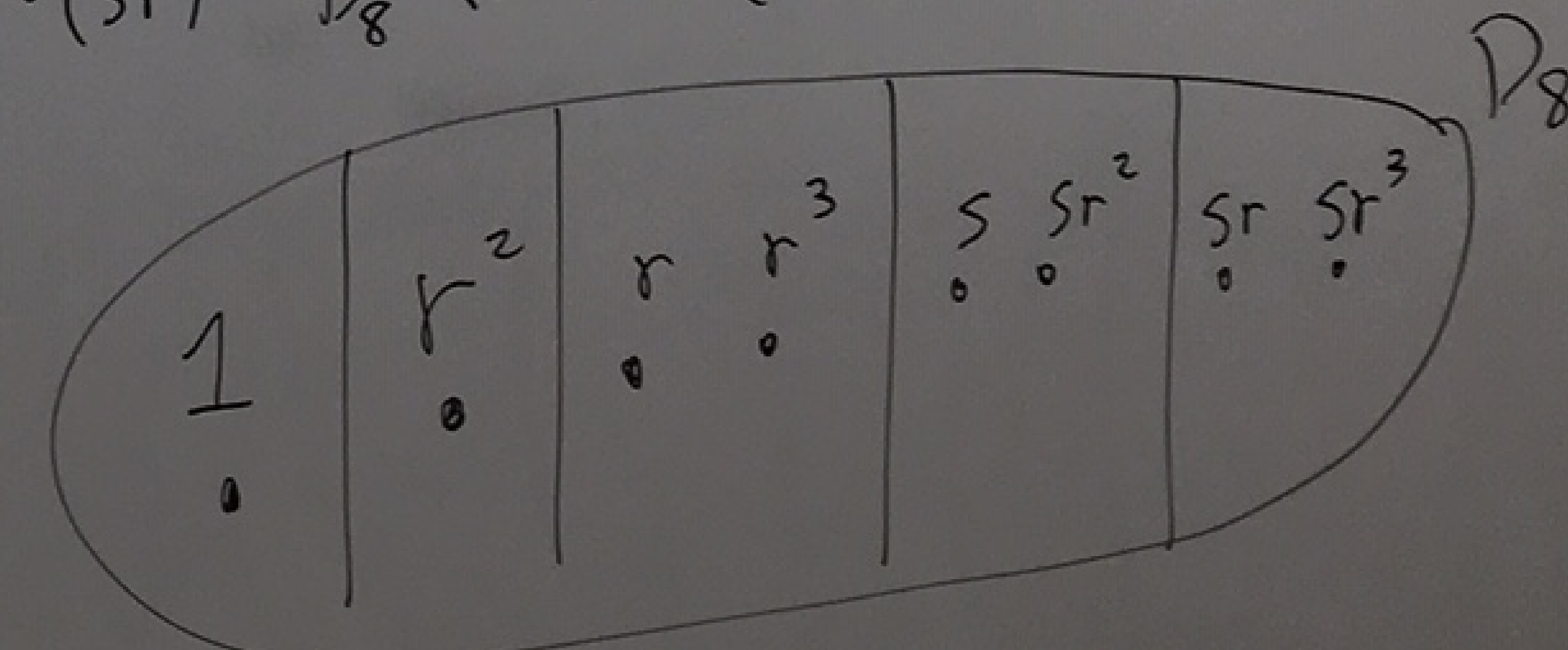
$D_8 \cdot r = D_8 \cdot r^3 = \{r, r^3\}$

$D_8 \cdot r^2 = \{r^2\}$

$D_8 \cdot s = D_8 \cdot (sr^2) = \{s, sr^2\}$

$D_8 \cdot (sr) = D_8 \cdot (sr^3) = \{sr, sr^3\}$

Conjugacy classes of D_8



HW
1.7
#18

Prop: Let G be a group acting on a non-empty set A.

The relation on A defined by $a \sim b$ iff $g \cdot a = b$ for some $g \in G$ is an equivalence relation on A.

The equivalence class of $a \in A$ is $G \cdot a$.

Furthermore,

$$|G \cdot a| = |G / G_a| = \frac{|G|}{|G_a|}$$

Ex: $\boxed{G = A = D_8 \mid g \cdot a = gag^{-1}}$

$$(D_8)_r = \{g \in D_8 \mid g \cdot r = r\}$$
$$= \{g \in D_8 \mid grg^{-1} = r\} = \{1, r, r^2, r^3\}$$

$|D_8 \cdot r| = |\{r, r^3\}| = 2$

$|D_8| / |(D_8)_r| = 8/4 = 2$

equal!

$D_8 \cdot$

proof of $|G\cdot a| = |G/G_a| = |G|/|G_a|$

Define $\psi : G\cdot a \longrightarrow G/G_a$
by $\psi(g\cdot a) = gG_a$.

ψ is well-defined

Suppose $g_1 \cdot a = g_2 \cdot a$ for some $g_1, g_2 \in G$.
We want to show that $\psi(g_1\cdot a) = \psi(g_2 \cdot a)$
We have $g_2^{-1}\cdot(g_1\cdot a) = g_2^{-1}\cdot(g_2\cdot a)$.
Then, $(g_2^{-1}g_1)\cdot a = (g_2^{-1}g_2)\cdot a$
So, $(g_2^{-1}g_1)\cdot a = a$.

So, $g_2^{-1}g_1 \in G_a$.
Thus, $g_2 G_a = g_1 G_a$.
Hence, $\psi(g_2\cdot a) = \psi(g_1 \cdot a)$.

ψ is 1-1

Suppose $\psi(g_1\cdot a) = \psi(g_2\cdot a)$ where $g_1, g_2 \in G$.
Then, $g_1 G_a = g_2 G_a$.
So, $g_2^{-1} g_1 \in G_a$.
Then $(g_2^{-1}g_1)\cdot a = a$.
So, $g_2 \cdot (g_2^{-1}g_1 \cdot a) = g_2 \cdot a$
Thus, $g_1 \cdot a = g_2 \cdot a$

ψ is onto

Let $gG_a \in G/G_a$ where $g \in G$.

Then, $g \cdot a \in G \cdot a$ and $\psi(g \cdot a) = gG_a$.

Since ψ is a bijection between $G \cdot a$ and G/G_a

we have that $|G| = |G/G_a|$.