10/22
Tuesday

$Z(G) \trianglelefteq G$ for any G.
So, $G/Z(G)$ is always a group.

$\boxed{3.1}$ 36 If $G/Z(G)$ is cyclic, then G is abelian.

<u>proof:</u> Since $G/Z(G)$ is cyclic,
$G/Z(G) = \langle gZ(G) \rangle$ for some $g \in G$.

Let $x, y \in G$.
We know $xZ(G) = [gZ(G)]^{\lambda} = g^{\lambda} Z(G)$
and $yZ(G) = [gZ(G)]^{\beta} = g^{\beta} Z(G)$ where $\lambda, \beta \in \mathbb{Z}$.

So, $x \in$
Thus, x
Check t
$xy =$

So G is a

elian.

So, $x \in g^\lambda Z(G)$ and $y \in g^\beta Z(G)$.

Thus, $x = g^\lambda z_1$ and $y = g^\beta z_2$ where $z_1, z_2 \in Z(G)$.

Check this out :

$$xy = g^\lambda z_1 g^\beta z_2 \underset{\boxed{z_1 \in Z(G)}}{=} g^\lambda g^\beta z_1 z_2 \underset{\boxed{z_2 \in Z(G)}}{=} g^\beta g^\lambda z_2 z_1 = g^\beta z_2 g^\lambda z_1 = yx.$$

So G is abelian.

$\square$

$,\beta \in \mathbb{Z}$.

Last time we proved this for abelian G

<u>Cauchy's Theorem</u>: If G is a finite group and p is a prime dividing $|G|$, then there exists $x \in G$ with $|x| = p$.

<u>proof:</u> We induct on $|G|$.

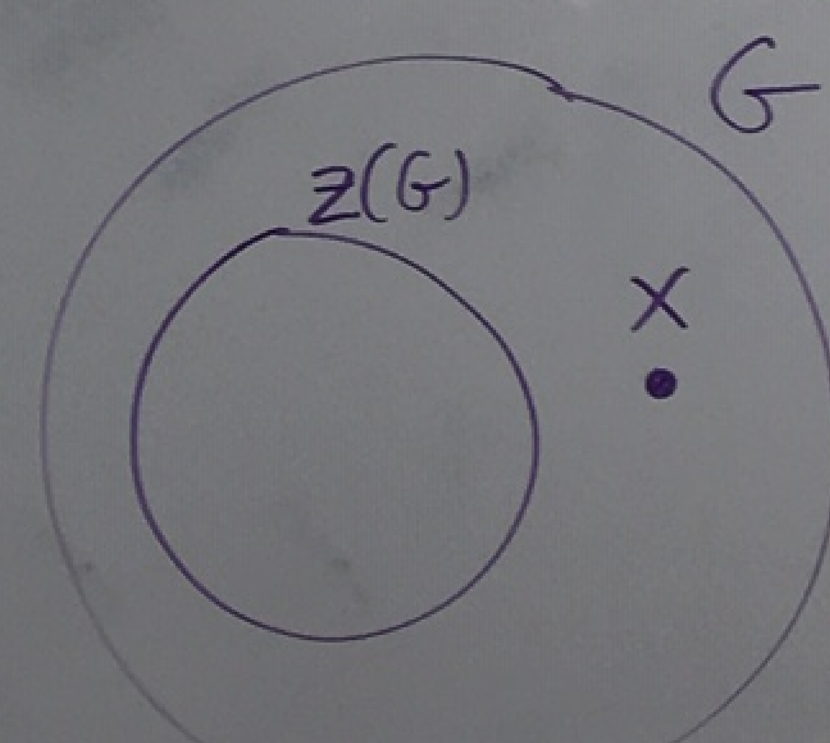
<u>base case:</u> Suppose $|G| = 2$.

Then $G = \{1, x\}$ with $x \neq 1$ and $x^2 = 1$.

So $|x| = 2$ and 2 is the only prime dividing $|G|$.

Suppose $|G|>2$ and the theorem is true for all groups of size smaller than $|G|$.

If G is abelian then by last class, the thm is true.
so assume G is not abelian.
Let p be a prime dividing $|G|$.
Since G is not abelian, $Z(G)\neq G$.



Suppose $x\in G$ with $x\notin Z(G)$.

The size of the conjugacy class of x is $\left|G/C_G(\{x\})\right| = \dfrac{|G|}{|C_G(\{x\})|}$.

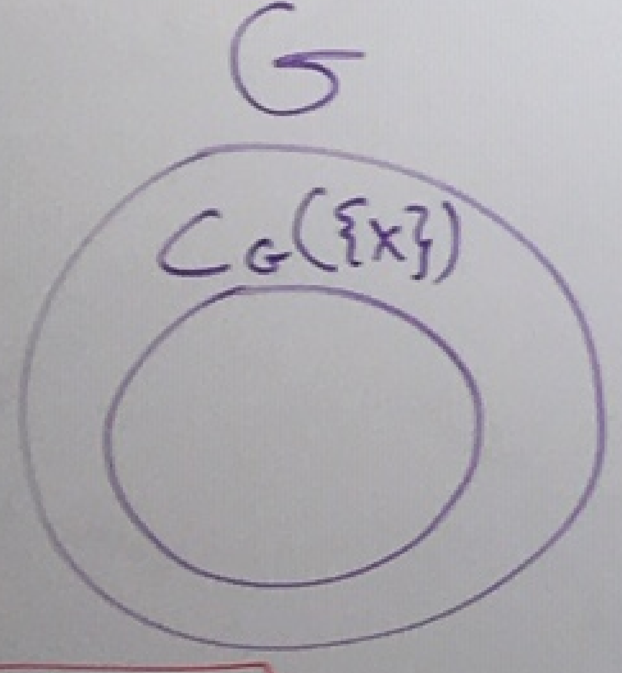
Since $x\notin Z(G)$, the conjugacy class of x is bigger than $\{x\}$. $\leftarrow$

So, $\dfrac{|G|}{|C_G(\{x\})|}>1$. $\leftarrow$ $|G|>|C_G(\{x\})|$.

Fact: Let C be the conjugacy class of x, then $C=\{x\}$ iff $x\in Z(G)$

pf:
If $x\in Z(G)$, then
$C=\{gxg^{-1}\mid g\in G\}=\{x\}$.
$gxg^{-1}=gg^{-1}x=x$

If $C=\{x\}$, then $gxg^{-1}=x$ for all $g\in G$. So $gx=xg$ for all $g\in G$.
That is $x\in Z(G)$. $\blacksquare$



$\boxed{a,b \in \mathbb{Z}}$

If p is a prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

If $p \mid |C_G(\{x\})|$, then since $|C_G(\{x\})| < |G|$ by the induction hypothesis there would be an element in $C_G(\{x\})$ of order p. So we'd have an element of G of order p since $C_G(\{x\}) \leq G$.

So we can assume that $p \nmid |C_G(\{x\})|$ for all $x \notin Z(G)$.

Therefore, since $|G| = \frac{|G|}{|C_G(\{x\})|} \cdot |C_G(\{x\})|$ we can assume that $p \mid \frac{|G|}{|C_G(\{x\})|}$ for all $x \notin Z(G)$.

Let $g_1, g_2, \dots, g_r$ be representatives of the distinct conjugacy classes not contained in $Z(G)$.

Then, by the class equation

$$|G| = |Z(G)| + \sum_{i=1}^{r} \frac{|G|}{|C_G(\{g_i\})|}.$$

So, p divides

$$|Z(G)| = |G| - \sum_{i=1}^{r} \frac{|G|}{|C_G(\{g_i\})|}.$$

p divides these

So, $p \mid |Z(G)|$.

Since $|Z(G)| < |G|$, by the induction hypothesis $Z(G)$ has an element of order p. And thus G does too. $\blacksquare$

Ex: Quaternion group

embed Q_8 into matrices over $\mathbb{C}$

matrix mult. is operation

$1 \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$-1 \longleftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad k \longleftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$i \longleftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\quad -k \longleftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$

$-i \longleftrightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

$j \longleftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$-j \longleftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$Q_8 = \{1, -1, i, -i, k, -k, j, -j\}$

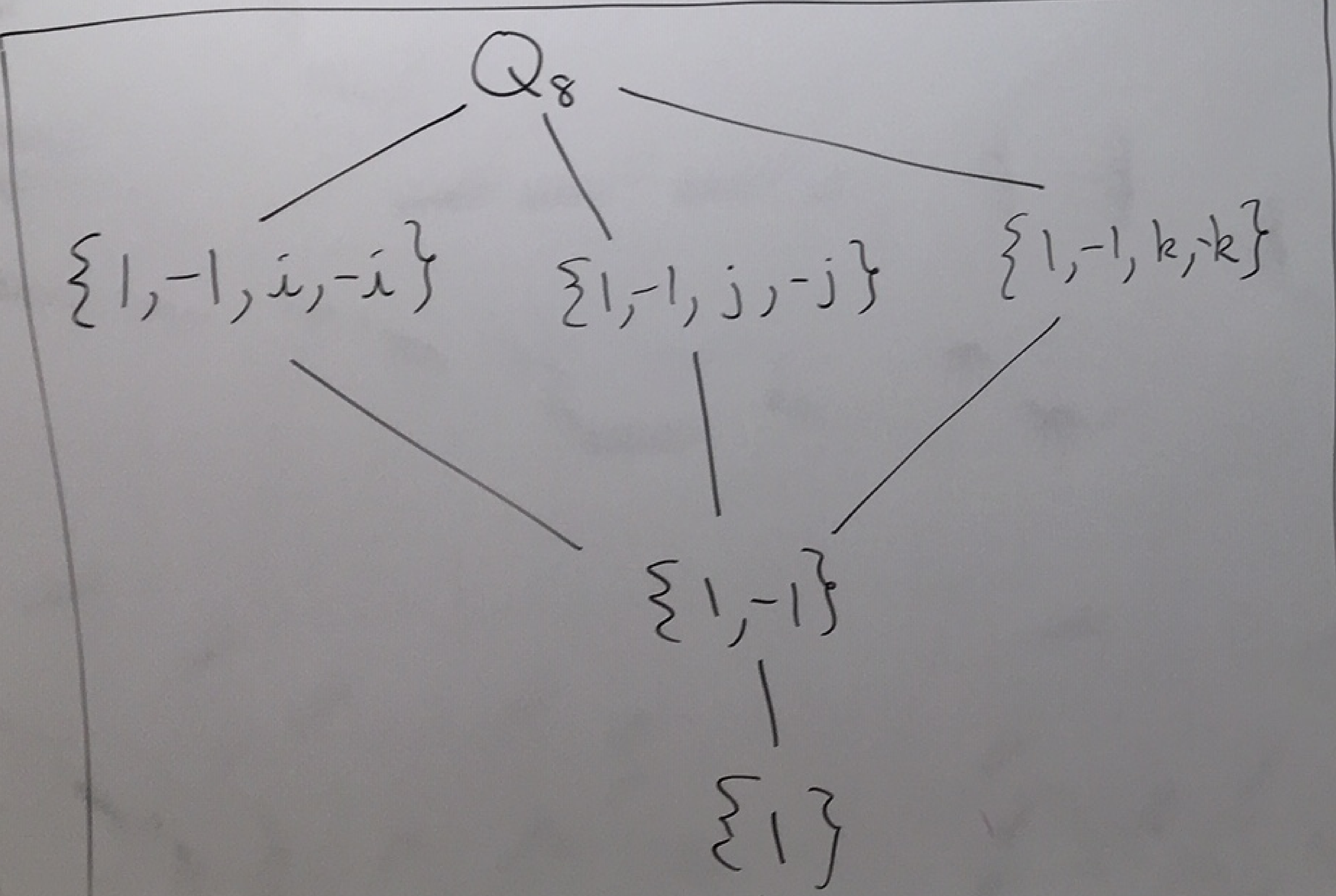
- $1 \cdot a = a \cdot 1 = a$ for all $a \in Q_8$
- $(-1) \cdot a = a(-1) = -a$ for all $a \in Q_8$
- $i^2 = j^2 = k^2 = -1$
- $ij = k \quad ji = -k$

  $jk = i \quad kj = -i$

  $ki = j \quad ik = -j$

Q_8 is not abelian.

Subgroups of Q_8



Q_8 is not abelian.

All the subgroups are cyclic (and abelian).

All the subgroups are normal subgroups.

$Q_8 \not\cong D_8$ since

Q_8

element	order
1	1
-1	2
$\pm i$	4
$\pm j$	4
$\pm k$	4

D_8

element	order
1	1
r	4
r^2	2
r^3	4
s	2
sr	2
sr^2	2
sr^3	2