12/5 Thursday

4.5

30 How many order 7 elements must a simple group of order $168 = 2^3 \cdot 3 \cdot 7$ have?

G is Simple means its only normal subgroups are $\{1\}$ and G

Let G be a group of size 168.

If $x \in G$ has order 7, then $|\langle x \rangle| = 7$.

So all the order 7 elements are contained in Sylow 7-subgroups.

...bgroups are $\{1\}$ and G

...mple

?

By Sylow's theorem $n_7 \equiv 1 \pmod 7$ and $n_7 \mid 2^3 \cdot 3$ ie $n_7 \mid 24$.

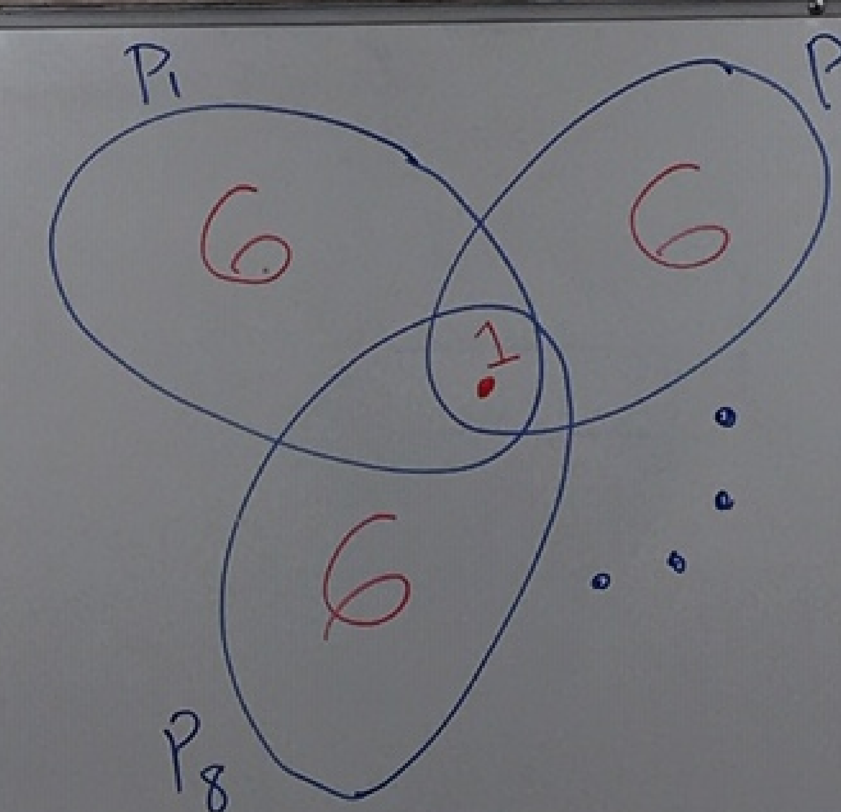
Divisors of 24 : 1, 2, 3, 4, 6, 8, 12, 24.

1 (mod 7)

So $n_7 = 1$ or $n_7 = 8$.

If $n_7 = 1$, then we would have a normal subgroup $P \trianglelefteq G$ of size 7. This contradicts G being simple.

Thus, $n_7 = 8$. So there exist exactly 8 subgroups of size 7. Call them $P_1, P_2, \dots, P_8$.



① Note if $i \neq j$ then $|P_i \cap P_j|$ divides $|P_i| = 7$.

So, $|P_i \cap P_j| = 1$ or $|P_i \cap P_j| = 7$.

If $|P_i \cap P_j| = 7$, then $P_i = P_j$.

So, $|P_i \cap P_j| = 1$.

That is $P_i \cap P_j = \{1\}$ if $i \neq j$.

② If $x \in P_i$ and $x \neq 1$, then $|x| = 7$ because $|x|$ divides $|P_i| = 7$.

By ① no order 7 elements are in two different P_i's.

So there are $6 \cdot 8 = 48$ elements of order 7.

Semi-direct product

$\text{Aut}(G) = \{ f : G \to G \mid f \text{ is an isomorphism} \}$

group operation is composition

Def: Let G and K be groups.

Let $\theta : K \to \text{Aut}(G)$ be a homomorphism.

Define an operation on $G \times K$ by

$$(g_1, k_1)(g_2, k_2) = (g_1 \underbrace{[\theta(k_1)]}_{\substack{\theta(k_1) : G \to G \\ \text{is an isomorphism}}}(g_2), k_1 k_2) = (g_1 \, {}^{k_1}g_2, k_1 k_2)$$

$$[\theta(k_1)](g_2) = {}^{k_1}g_2$$

$G \times K$ equipped with this operation is called the <u>semi-direct product of G and K with respect to θ</u> and is denoted by $G \rtimes_\theta K$ or $G \rtimes K$.

Note:

$$ {}^{k_1k_2}g = \left(\Theta(k_1k_2)\right)(g) = \left(\Theta(k_1)\circ\Theta(k_2)\right)(g) = [\Theta(k_1)]\left([\Theta(k_2)](g)\right) $$

$$ = [\Theta(k_1)]\left[{}^{k_2}g\right] = {}^{k_1}\left({}^{k_2}g\right) $$

$$ {}^{k}(g_1g_2) = \underbrace{[\Theta(k)]}_{\substack{\text{isomorphism} \\ \Theta(k):G\to G}}(g_1g_2) = [\Theta(k)](g_1)\cdot[\Theta(k)](g_2) = \left({}^{k}g_1\right)\left({}^{k}g_2\right) $$

Thm: $G \rtimes K$ is a group.

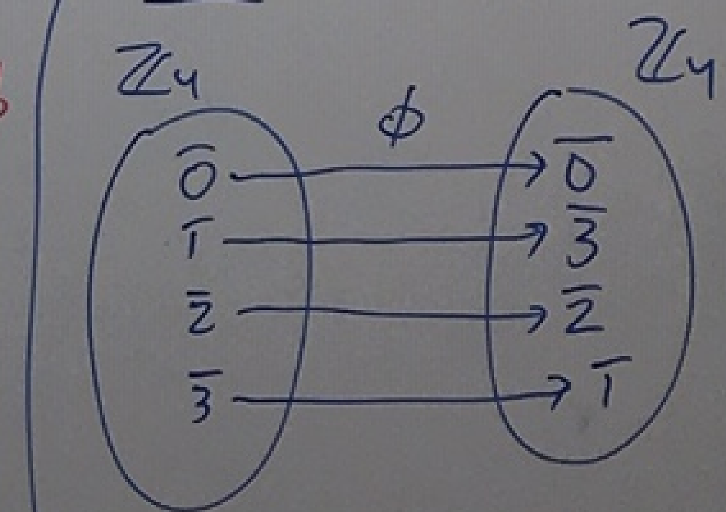
Ex: We are going to construct D_{2n}.

Let $\phi: \mathbb{Z}_n \to \mathbb{Z}_n$ by $\phi(\bar{x}) = \overline{-x}$.

Then $\phi \in Aut(\mathbb{Z}_n)$

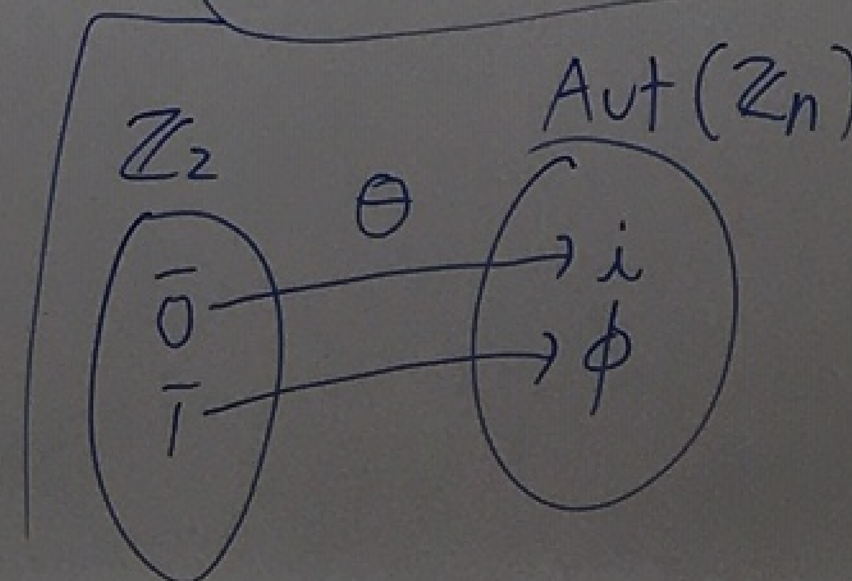
Note $\phi^2 = i$ where $i: \mathbb{Z}_n \to \mathbb{Z}_n$ is the identity function $i(\bar{x}) = \bar{x}$ $\Big\}\ \phi^2 = \phi\circ\phi$

Ex: $n=4$



Define $\Theta: \mathbb{Z}_2 \to Aut(\mathbb{Z}_n)$ by

$$ \Theta(\bar{b}) = \begin{cases} i, & \text{if } \bar{b} = \bar{0} \\ \phi, & \text{if } \bar{b} = \bar{1} \end{cases} $$



Θ is a homomorphism

Let $G = \mathbb{Z}_n \rtimes_\theta \mathbb{Z}_2 = \mathbb{Z}_n \rtimes \mathbb{Z}_2$.

It turns out that $G \cong D_{2n}$.

$$\boxed{(g_1, k_1)(g_2, k_2) = (g_1 \, {}^{k_1}g_2, k_1 k_2)}$$

<u>Ex:</u> $(n = 3)$

identity

$$G = \mathbb{Z}_n \rtimes \mathbb{Z}_2 = \{\boxed{(\bar{0}, \bar{0})}, (\bar{1}, \bar{0}), (\bar{2}, \bar{0}), (\bar{0}, \bar{1}), (\bar{1}, \bar{1}), (\bar{2}, \bar{1})\}$$

$$(\bar{1}, \bar{0})(\bar{1}, \bar{1}) = (\bar{1} + {}^{\bar{0}}\bar{1}, \bar{0} + \bar{1}) = (\bar{1} + \bar{1}, \bar{1}) = (\bar{2}, \bar{1}) \leftarrow {}^{\bar{0}}\bar{1} = [\theta(\bar{0})](\bar{1}) = i(\bar{1}) = \bar{1}$$

$\boxed{\text{order of } (\bar{1}, \bar{0}) \text{ is order 3}}$

$$(\bar{1}, \bar{0}) \neq (\bar{0}, \bar{0})$$

$$(\bar{1}, \bar{0})(\bar{1}, \bar{0}) = (\bar{1} + {}^{\bar{0}}\bar{1}, \bar{0} + \bar{0}) = (\bar{2}, \bar{0}) \neq (\bar{0}, \bar{0})$$

$$(\bar{1}, \bar{0})(\bar{1}, \bar{0})(\bar{1}, \bar{0}) = (\bar{2}, \bar{0})(\bar{1}, \bar{0}) = (\bar{2} + {}^{\bar{0}}\bar{1}, \bar{0} + \bar{0}) = (\bar{2} + \bar{1}, \bar{0}) = (\bar{3}, \bar{0}) = (\bar{0}, \bar{0})$$

Not

$k_1 k_2$

${}^{k}(g$

<u>Thm:</u>

$\bar{1}) = \lambda(\bar{1}) = \bar{1}$

<u>Denote:</u>

$$1 = (\bar{0}, \bar{0})$$
$$\hat{r} = (\bar{1}, \bar{0})$$
$$\hat{s} = (\bar{0}, \bar{1})$$

Then: $\hat{r}^2 = (\bar{2}, \bar{0})$

$$\hat{s}\hat{r} = (\bar{0},\bar{1})(\bar{1},\bar{0}) = (\bar{0} +^{\bar{1}} \bar{1}, \bar{1}+\bar{0}) = (\bar{0}+\bar{2}, \bar{1}) = (\bar{2}, \bar{1})$$

$$\hat{s}\hat{r}^2 = (\bar{0},\bar{1})(\bar{2},\bar{0}) = (\bar{0} +^{\bar{1}} \bar{2}, \bar{1}+\bar{0}) = (-\bar{2}, \bar{1}) = (\bar{1}, \bar{1})$$

$$\hat{r}\hat{s} = (\bar{1},\bar{0})(\bar{0},\bar{1}) = (\bar{1} +^{\bar{0}} \bar{0}, \bar{1}) = (\bar{1}, \bar{1}) = \hat{s}\hat{r}^2$$

$$\begin{array}{ccccccccc} G = \{ & \hat{1}, & \hat{r}, & \hat{r}^2, & \hat{s}, & \hat{s}\hat{r}, & \hat{s}\hat{r}^2 & \} \\ & \psi \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ D_{2n} = \{ & 1, & r, & r^2, & s, & sr, & sr^2 & \} \end{array}$$

Define $\psi : G \to D_{2n}$ as above. You can check that ψ is an isomorphism.

$\boxed{\text{Or } \psi(\bar{a}, \bar{b}) = r^a s^b}$

So, $G \cong D_{2n}$.