Strong Edge-Coloring for Cubic Halin Graphs

Gerard Jennhwa Chang[123*] and Daphne Der-Fen Liu[4†]

[1]Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan

[2]Taida Institute for Mathematical Sciences, National Taiwan University, Taipei 10617, Taiwan

[3]National Center for Theoretical Sciences, Taipei Office, Taipei, Taiwan

[4]Department of Mathematics, California State University, Los Angeles, USA

November 4, 2011 (revision January 11, 2012)

Abstract

A strong edge-coloring of a graph G is a function that assigns to each edge a color such that two edges within distance two apart must receive different colors. The minimum number of colors used in a strong edge-coloring is the *strong chromatic index* of G. Lih and Liu [14] proved that the strong chromatic index of a cubic Halin graph, other than two special graphs, is 6 or 7. It remains an open problem to determine which of such graphs have strong chromatic index 6. Our article is devoted to this open problem. In particular, we disprove a conjecture of Shiu, Lam and Tam [18] that the strong chromatic index of a cubic Halin graph with characteristic tree a caterpillar of odd leaves is 6.

1 Introduction

The coloring problem considered in this article has restrictions on edges within distance two apart. The *distance* between two edges e and e' in a graph is the minimum

*E-mail: gjchang@math.ntu.edu.tw. Supported in part by the National Science Council under grant NSC98-2115-M-002-013-MY3.

†Corresponding author. Email: dliu@calstatela.edu.

k for which there is a sequence $e_0, e_1, \ldots, e_k$ of distinct edges such that $e = e_0$, $e' = e_k$, and e_{i-1} shares an end vertex with e_i for $1 \leq i \leq k$. A *strong edge-coloring* of a graph is a function that assigns to each edge a color such that any two edges within distance two apart must receive different colors. A *strong k-edge-coloring* is a strong edge-coloring using at most k colors. The *strong chromatic index* of a graph G, denoted by $\chi'_s(G)$, is the minimum k such that G admits a strong k-edge-coloring.

Strong edge-coloring was first studied by Fouquet and Jolivet [8, 9] for cubic planar graphs. A trivial upper bound is that $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$ for any graph G of maximum degree Δ. Fouquet and Jolivet [8] established a Brooks type upper bound $\chi'_s(G) \leq 2\Delta^2 - 2\Delta$, which is not true only for $G = C_5$ as pointed out by Shiu and Tam [19]. The following conjecture was posed by Erdős and Nešetřil [5, 6] and revised by Faudree, Schelp, Gyárfás and Tuza [7]:

Conjecture 1. *For any graph G of maximum degree Δ,*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{if } \Delta \text{ is odd.} \end{cases}$$

Faudree, Schelp, Gyárfás and Tuza [7] also asked whether $\chi'_s(G) \leq 9$ if G is cubic planar. If this upper bound is proved to be true, it would be the best possible. For graphs with maximum degree $\Delta = 3$, Conjecture 1 was verified by Andersen [1] and by Horák, Qing and Trotter [12] independently. For $\Delta = 4$, while Conjecture 1 says that $\chi'_s(G) \leq 20$, Horák [11] obtained $\chi'_s(G) \leq 23$ and Cranston [4] proved $\chi'_s(G) \leq 22$.

The main theme of this paper is to study strong edge-coloring for the following planar graphs. A *Halin graph* $G = T \cup C$ is a plane graph consisting of a plane embedding of a tree T each of whose interior vertex has degree at least 3, and a cycle C connecting the leaves (vertices of degree 1) of T such that C is the boundary of the exterior face. The tree T and the cycle C are called the *characteristic tree* and the *adjoint cycle* of G, respectively. Strong chromatic index for Halin graphs was first considered by Shiu, Lam and Tam [18] and then studied in [19, 13, 14].

A *caterpillar* is a tree whose removal of leaves results in a path called the *spine* of the caterpillar. For $k \geq 1$, let $\mathcal{G}_k$ be the set of all cubic Halin graphs whose characteristic trees are caterpillars with $k + 2$ leaves. For a graph $G = T \cup C$ in $\mathcal{G}_k$, let $P\colon v_1, v_2, \ldots, v_k$ be the spine of T and each v_i is adjacent to a leaf u_i for $1 \leq i \leq k$ with v_1 (resp. v_k) adjacent to one more leaf $u_0 = v_0$ (resp. $u_{k+1} = v_{k+1}$). We draw G on the plane by putting the path $v_0 P v_{k+1}$ horizontally in the middle, and the pending edges (leaf edges) v_iu_i, $1 \le i \le k$, by either up or down edges vertically to P. See Figure 1 for an example of $\mathcal{G}_8$.

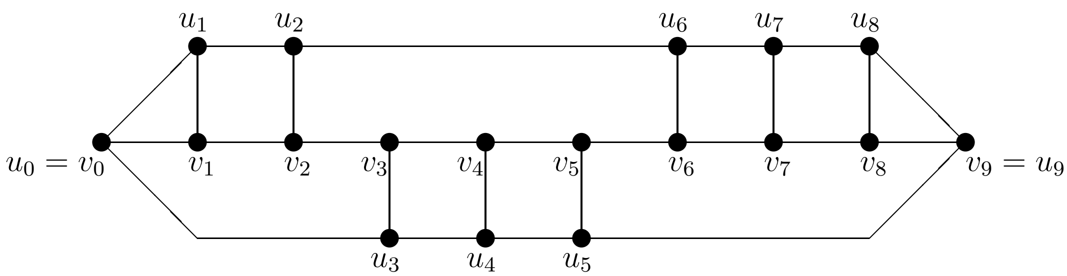

Figure 1: The graph $G_{2,3,3}$ in $\mathcal{G}_8$.

From this drawing, we associate G with a list of positive integers $(n_1, n_2, \dots, n_r)$, where n_i is the number of maximum consecutive up or down edges, starting from the leftmost to the rightmost on P. We use $G_{n_1,n_2,\dots,n_r}$ to denote this graph. For instance the graph in Figure 1 is $G_{2,3,3}$. Notice that $n_1 + n_2 + \cdots + n_r = k$. For a special case when these pending edges are all in the same direction (up or down), the graph G_k is called the *necklace* and denoted by Ne_k in [18]. Notice that G_k is the only graph in $\mathcal{G}_k$ for $k \le 3$.

Observation 1. $G_{n_1,n_2,\dots,n_r} \cong G_{n_r,\dots,n_2,n_1}$.

Observation 2. $G_{n_1,n_2,\dots,n_r,1} \cong G_{n_1,n_2,\dots,n_r+1}$.

It is easy to see that $\chi'_s(G) \ge 6$ for any $G \in \mathcal{G}_k$, $k \ge 1$. Shiu, Lam and Tam [18] obtained the following results:

- $$\chi'_s(G_k) = \begin{cases} 9, & k = 2; \\ 8, & k = 4; \\ 7, & k \text{ is even and } k \ge 6; \\ 6, & k \text{ is odd.} \end{cases}$$
- If $G \in \mathcal{G}_k$ with $k \ge 4$, then $6 \le \chi'_s(G) \le 8$.
- If G is a cubic Halin graph, then $6 \le \chi'_s(G) \le 9$.

Moreover, the authors [18] raised the following conjectures:

Conjecture 2. *If $G \in \mathcal{G}_k$ with $k \ge 5$, then $\chi'_s(G) \le 7$.*

Conjecture 3. *If $G \in \mathcal{G}_k$ with odd $k \geq 5$, then $\chi'_s(G) = 6$.*

Conjecture 4. *If $G = T \cup C$ is a Halin graph, then $\chi'_s(G) \leq \chi'_s(T) + 4$.*

Faudree, Schelp, Gyárfás and Tuza [7] proved, for any tree T, it holds that $\chi'_s(T) = \max_{uv \in E(T)}(\deg(u) + \deg(v) - 1)$. Conjecture 4 was confirmed by Lai, Lih and Tsai [13], who proved a stronger result that $\chi'_s(G) \leq \chi'_s(T) + 3$ for any Halin graph $G = T \cup C$ other than G_2 and wheels W_n with $n \not\equiv 0 \pmod 3$, where $W_n = K_{1,n} \cup C_n$. Note that $\chi'_s(W_5) = \chi'_s(K_{1,5}) + 5$; and $\chi'_s(G) = \chi'_s(T) + 4$ for $G = G_2$ or $G = W_n$ with $n \not\equiv 0 \pmod 3$ and $n \neq 5$.

Conjecture 2 was confirmed by Lih and Liu [14], who proved a more general result that $\chi'_s(G) \leq 7$ is true for any cubic Halin graph other than G_2 and G_4. Hence, the strong chromatic index for any cubic Halin graph $G \neq G_2, G_4$ is either 6 or 7.

It remains open to determine the cubic Halin graphs G with $\chi'_s(G) = 6$ (or the ones with $\chi'_s(G) = 7$). Our aim is to investigate this problem. In particular, we establish methods that can be used to study the graphs $\mathcal{G}_k$. As a result, we discover counterexamples to Conjecture 3. We prove that for any $k \geq 7$, there exists graph $G \in \mathcal{G}_k$ with $\chi'_s(G) = 7$; and for any $k \neq 2, 4$, there exists $G \in \mathcal{G}_k$ (other than necklaces) with $\chi'_s(G) = 6$. In Section 4, we determine the value of $\chi'_s(G)$ for some special families of graphs G in $\mathcal{G}_k$.

2 Cubic Halin graphs G with $\chi'_s(G) = 6$

This section gives some cubic Halin graphs with strong chromatic index 6. We begin with the development of several general transformation theorems for Halin graphs.

For a positive integer r, an *r-tail* of a tree T is a path $P_r\colon v_1, v_2, \ldots, v_r, v_{r+1}$ in which v_1 is not a leaf but all vertices in $L_i = \{u \notin P\colon uv_i \in E(T)\}$ are leaves for $1 \leq i \leq r$. For integer $s < r$, *cutting* P_s from T means deleting the vertices $\{v_1, v_2, \ldots, v_{s-1}\} \cup_{1 \leq i \leq s} L_i$ from T, which results in a tree denoted by $T \ominus P_s$. Notice that v_s becomes a leaf adjacent to v_{s+1} in $T \ominus P_s$.

Suppose $P\colon v_1, v_2, \ldots, v_r, v_{r+1}$ is an r-tail of the characteristic tree T of a Halin graph $G = T \cup C$. For any j with $1 \leq j \leq r$, the vertices in $\cup_{1 \leq i \leq j} L_i$ form a consecutive portion on the adjoint cycle C. See Figure 2 for an example of a 4-tail. For any two vertices in $\cup_{2 \leq i \leq r} L_i$, we may regard that they are on the *same* or *different* sides of L_1. For instance, in Figure 2, u_3^1 and u_3^2 are on the same side of L_1, while u_2^1

and u_2^2 are on different sides of L_1. For $s < r$, the tree $T \ominus P_s$ is the characteristic tree of a new Halin graph, denoted by $G \ominus P_s$, whose adjoint cycle is obtained from C by replacing the segment $\{x\} \cup_{1 \le i \le s} L_i \cup \{y\}$ by the path xv_sy originally not in G, where x (respectively, y) is the vertex in C right before (respectively, after) $\cup_{1 \le i \le s} L_i$. See the dashed path for xv_3y in Figure 2.

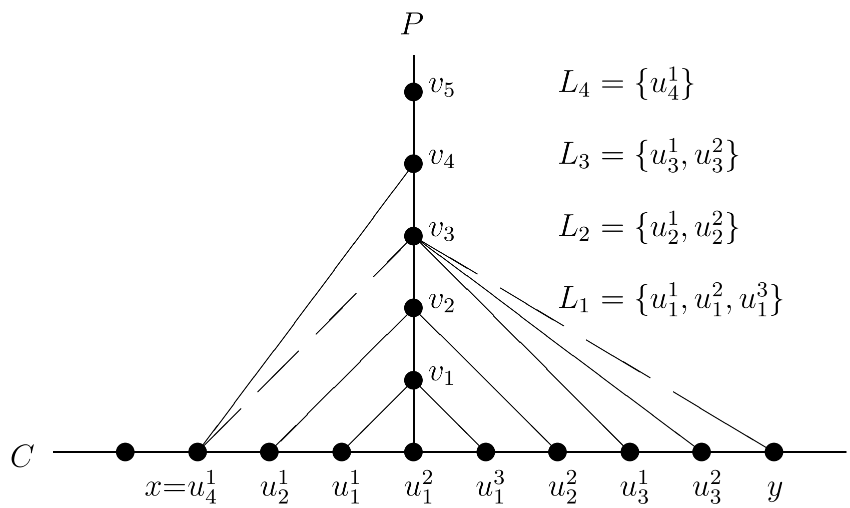

Figure 2: A cutting 4-tail from T, resulting in $G \ominus P_4$ with two new edges, v_3x and v_3y, while vertices in $\{v_1, v_2\} \cup L_1 \cup L_2 \cup L_3$ are all gone.

We denote a 4-cycle by (x_1, x_2, x_3, x_4), which consists of the edges x_4x_1, and x_ix_{i+1} for $i = 1, 2, 3$.

Lemma 3. *Suppose* (x_1, x_2, x_3, x_4) *is a* 4*-cycle in a graph* G *in which each* x_i *is adjacent to a vertex* y_i *not in the* 4*-cycle for* $1 \le i \le 4$. *If* $\chi'_s(G) = 6$, *then for every strong* 6*-edge-coloring* f *of* G *we have*

(i) $f(x_1y_1) = f(x_3y_3)$ *and* $f(x_2y_2) = f(x_4y_4)$, *and*

(ii) $f(y_3y_4) = f(x_1x_2)$ *whenever* y_3 *is adjacent to* y_4.

Proof. Part (i) follows from that for each i the edges on the 4-cycle (x_1, x_2, x_3, x_4) together with the edges x_iy_i and $x_{i+1}y_{i+1}$ use all the 6 colors, where $x_5y_5 = x_1y_1$.

Part (ii) follows from that the edges on the 4-cycle (x_3, y_3, y_4, x_4) together with the two edges x_1x_4, x_2x_3 use all the 6 colors. See Figure 3 for an illustration. □

We now consider the cutting tail operation for the characteristic tree of a cubic Halin graph $G = T \cup C$. We shall study the conditions for which such an operation preserves the fact that $\chi'_s(G) = 6$.

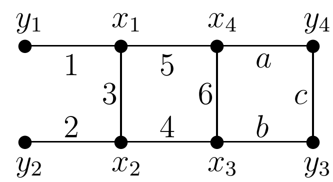

Figure 3: a, b, c are forced to be $2, 1, 3$, respectively.

Theorem 4. *Suppose $P: v_1, v_2, v_3, v_4$ is a 4-tail of the characteristic tree T of a cubic Halin graph $G = T \cup C$, where $L_1 = \{u_0, u_1\}$ and $L_i = \{u_i\}$ for $i \geq 2$. If u_2 and u_3 are on the same side of L_1, then $\chi'_s(G) = 6$ if and only if $\chi'_s(G \ominus P_2) = 6$.*

Proof. ($\Rightarrow$) Suppose $\chi'_s(G) = 6$. Let f be a strong 6-edge-coloring of G. Without loss of generality, we may assume that $f(xu_0) = 1$, $f(u_0u_1) = 2$, $f(u_1u_2) = 3$, $f(v_1u_0) = 4$, $f(v_1u_1) = 5$, and $f(v_1v_2) = 6$ as the bold faced numbers in Figure 4. It is then the case that $f(v_2u_2) = 1$. Repeatedly applying Lemma 3, we have $f(u_2u_3) = 4$, $f(v_2v_3) = 2$, $f(v_3u_3) = 5$, $f(u_3z) = 6$ and $f(v_3v_4) = 3$ (see Figure 4). In $G \ominus P_2$, we use the old color for edges in G, and color the new edges xv_2 and v_2y by 1 and 4, respectively. It is easy to check that the new coloring is a strong 6-edge-coloring for $G \ominus P_2$. Hence, $\chi'_s(G \ominus P_2) = 6$.

($\Leftarrow$) Suppose $\chi'_s(G \ominus P_2) = 6$. Let f' be a strong 6-edge-coloring of $G \ominus P_2$. Without loss of generality, assume that the colors are as in Figure 4. We may delete the edges xv_2 and v_2y, and extend the coloring to G using the colors as in Figure 4. This gives a strong 6-edge-coloring of G, so $\chi'_s(G) = 6$. $\square$

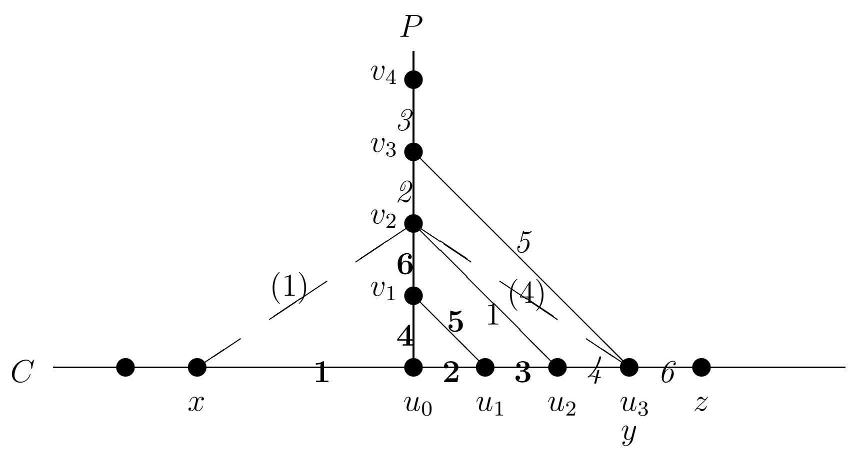

Figure 4: A cutting $G \ominus P_2$.

Corollary 5. *Suppose* $n_1 + n_2 + \cdots + n_r \geq 2$*. Then* $\chi'_s(G_{n_1,n_2,\ldots,n_r}) = 6$ *if and only if* $\chi'_s(G_{n_1,n_2,\ldots,n_r+2}) = 6$*.*

Theorem 6. *Suppose* $P\colon v_1, v_2, v_3, v_4, v_5$ *is a 5-tail of the characteristic tree* T *of a cubic Halin graph* $G = T \cup C$*, where* $L_1 = \{u_0, u_1\}$ *and* $L_i = \{u_i\}$ *for* $i \geq 2$*. Assume* u_2 *and* u_3 *are on different sides of* L_1*, while* u_2 *and* u_4 *are on the same side of* L_1*. If* $\chi'_s(G \ominus P_2) = 6$*, then* $\chi'_s(G) = 6$*.*

Proof. Let f' be a strong 6-edge-coloring of $G \ominus P_2$. By Lemma 3, without loss of generality, we may assume that $f'(v_3v_4) = 1$, $f'(v_2v_3) = 2$, $f'(v_2y) = 3$, $f'(v_4u_4) = f'(wx) = 4$, $f'(v_4v_5) = f'(xv_2) = 5$ and $f'(v_3u_3) = f'(u_4z) = 6$, as the bold faced numbers shown in Figure 5. We delete the edges xv_2 and v_2y from $G \ominus P_2$, and extend the coloring to G using the colors shown in Figure 5. This gives a strong 6-edge-coloring of G, so $\chi'_s(G) = 6$. □

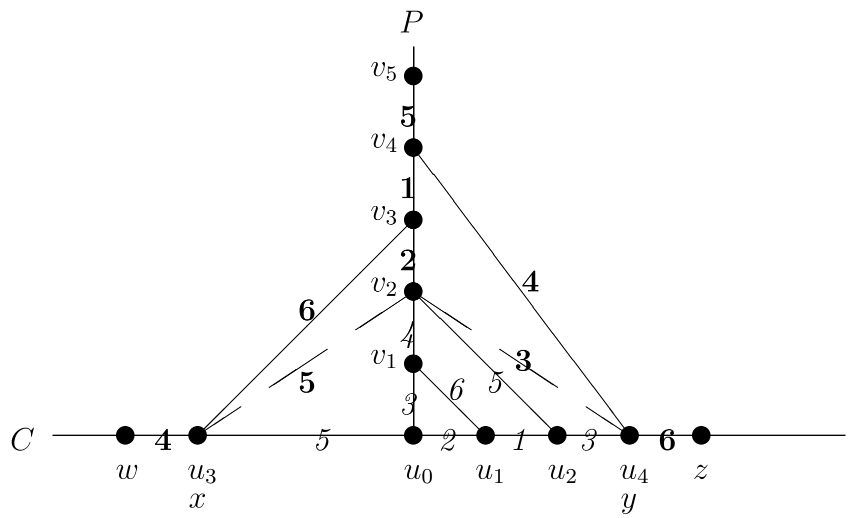

Figure 5: A cutting $G \ominus P_2$.

Remark that unlike Theorem 4 the converse of Theorem 6 is not true. This can be seen by the example given blow that $\chi'_s(G_{1\star 8}) = \chi'_s(G_{1\star 6,2}) = 6$, while $\chi'_s(G_{1\star 6}) = 7$. (See Corollary 11.)

Corollary 7. *If* $n_1+n_2+\cdots+n_r \geq 2$ *and* $\chi'_s(G_{n_1,n_2,\ldots,n_r,1}) = 6$*, then* $\chi'_s(G_{n_1,n_2,\ldots,n_r,1,2}) = 6$*.*

Corollary 8. *Assume* $\chi'_s(G_{n_1,n_2,\ldots,n_{r-1},1}) = 6$ *where* $n_1 + n_2 + \cdots + n_{r-1} \geq 2$*. Then* $\chi'_s(G_{n_1,n_2,\ldots,n_{r-1},F_1,F_2,\ldots,F_k,1}) = 6$*, where each* F_i *is either a single positive even integer, or a list of two integers,* $(1, t)$*, for some odd integer* t*. In particular,* $\chi'_s(G_{n_1,n_2,\ldots,n_{r-1},1\star m}) = 6$ *for odd* m*, where* $1 \star m$ *stands for a sequence of* m *1's.*

Proof. Assume $\chi'_s(G_{n_1,n_2,\ldots,n_{r-1},1}) = 6$. It suffices to prove the result for the case $k = 1$. Assume F_1 is a single even integer, $F_1 = n_r$. By Corollary 5 and Observation 2,

$$6 = \chi'_s(G_{n_1,n_2,\ldots,n_{r-1},1}) = \chi'_s(G_{n_1,n_2,\ldots,n_{r-1},1+n_r}) = \chi'_s(G_{n_1,n_2,\ldots,n_{r-1},n_r,1}).$$

Next, assume F_1 is a list of two integers $(1, t)$ for some odd $t = 2s + 1$. By Corollaries 7 and 5, and Observation 2,

$$6 = \chi'_s(G_{n_1,n_2,\ldots,n_{r-1},1,2}) = \chi'_s(G_{n_1,n_2,\ldots,n_{r-1},1,2(s+1)}) = \chi'_s(G_{n_1,n_2,\ldots,n_{r-1},1,t,1}).$$

□

Let k be an even integer. Although it is known [18] that $\chi'_s(G_k) > 6$, there exist graphs $G \in \mathcal{G}_k$ with $\chi'_s(G) = 6$. Figures 6 and 7 show two examples. For positive integers x and n, we denote $x \star n$ as an n-term repeated sequence of x.

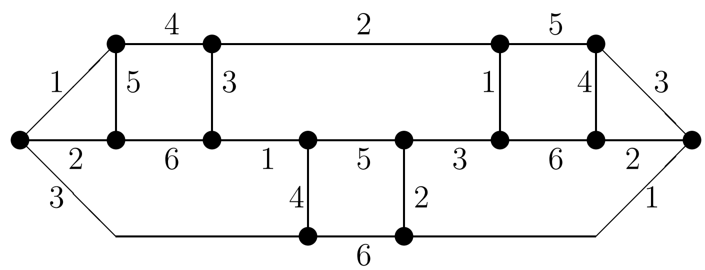

Figure 6: A strong 6-edge-coloring for $G_{2,2,2} = G_{2,2,1,1}$.

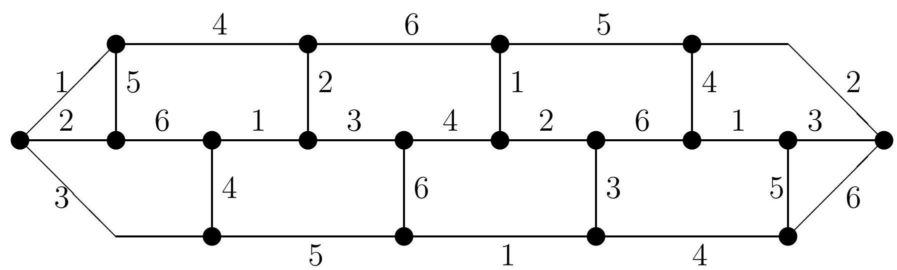

Figure 7: A strong 6-edge-coloring for $G_{1\star 8}$.

By the results we have shown, one can verify that for every positive integer $k \neq 2, 4$, there exists $G \in \mathcal{G}_k$ (other than necklaces) with $\chi'_s(G) = 6$. This is because $\chi'_s(G_1) = \chi'_s(G_{1,1,1}) = \chi'_s(G_{2,2,1,1}) = \chi'_s(G_{1\star 8}) = 6$, by Corollary 8, one gets $\chi'_s(G_{2,2,1\star m}) = 6$ for even $m \geq 4$, and $\chi'_s(G_{1\star n}) = 6$ for $n \neq 2, 4, 6$.

3 Cubic Halin graphs G with $\chi'_s(G) = 7$

We present some cubic Halin graphs with strong chromatic index 7. In particular, we prove that for any $k \geq 7$, there exists $G \in \mathcal{G}_k$ with $\chi'_s(G) = 7$.

Let us start with an example, $\chi'_s(G_{2,2}) = 7$. Suppose to the contrary that $\chi'_s(G_{2,2}) = 6$. Choose a strong 6-edge-coloring f for $G_{2,2}$. By Lemma 3 (i), $f(v_0u_3) = f(v_4v_5)$. Since the color $f(v_0u_3)$ has to be used by the 4-cycle (u_1, u_2, v_2, v_1), it is the case that $f(v_0u_3) = f(u_2v_2)$, and so $f(v_4v_5) = f(u_2v_2)$, a contradiction.

Lemma 9. *Let $G = G_{2,3,1,n_4,n_5\dots,n_r}$ with $r \geq 4$. If f is a strong 6-edge-coloring for G, then $f(u_1v_1) = f(u_6u_{7+n_4}) = f(v_7v_8)$.*

Proof. Without loss of generality, assume that $f(v_0u_1) = 1$, $f(v_0v_1) = 2$, $f(v_0u_3) = 3$, $f(v_1u_1) = 4$, $f(u_1u_2) = 5$, and $f(v_1v_2) = 6$. See the bold faced numbers in Figure 8. Then $f(v_2u_2) = 3$. By Lemma 3 (i), $f(u_2u_6) = 2$, $f(v_2v_3) = f(u_4u_5) = 1$, and $f(v_4v_5) = 3$. See the italic numbers in Figure 8.

Suppose $f(v_3u_3) = x$ and $f(u_3u_4) = y$. Then $x \in \{4, 5\}$ and $y \in \{4, 5, 6\}$. By Lemma 3, $f(v_5u_5) = x$ and $f(v_5v_6) = y$. Let z be the only label in $\{4, 5, 6\} - \{x, y\}$. Then $\{f(v_6v_7), f(v_6u_6)\} = \{1, z\}$, since v_6v_7 and v_6u_6 cannot be labeled by $2, 3, x, y$. Let $f(u_6u_{7+n_4}) = a$. Then $a \notin \{1, 2, 3, 5, y, z\}$. Hence, it must be the case that $a = x \neq 5$, implying $a = x = 4$.

By Lemma 3 (i), $f(u_5u_7) = f(v_3v_4) = c \in \{5, 2\}$. If $c = 5$, then $f(v_6v_7) = 1$ and $f(v_6u_6) = z = c = 5$, which is impossible as $f(u_1u_2) = 5$. Hence, $c = 2$. Then $f(v_7u_7) \notin \{1, 2, x, y, z\}$, so $f(v_7u_7) = 3$. Consequently, $f(v_7v_8) = b \notin \{1, 2, 3, y, z\}$. Therefore, $b = x = 4$. This completes the proof. ☐

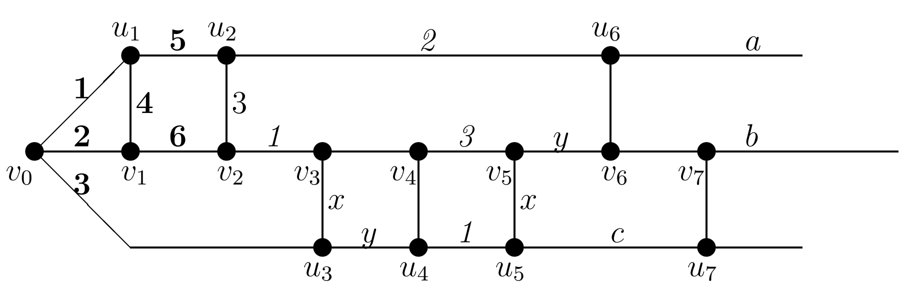

Figure 8: In $G_{2,3,1,n_4,n_5,\dots,n_r}$, labels a and b are forced to be 4.

Theorem 10. *The following graphs have strong chromatic index* 7*:*

(a) $G_{2,3,1,n_4}$.

(b) $G_{2,3,1,1,n_5,n_6,\ldots,n_r}$ *with* $r \geq 5$.

(c) $G_{2,3,1,3,n_5}$.

(d) $G_{2,3,1,3,2,n_6,n_7,\ldots,n_r}$ *with* $r \geq 6$.

(e) $G_{2,3,1,3,4,n_6}$.

(f) $G_{2,3,1,3,4,2,n_7,n_8,\ldots,n_r}$ *with* $r \geq 7$.

Proof. For each case in the following, we suppose to the contrary that the given graph has strong chromatic index 6. Let f be a strong 6-edge-coloring for G. We shall derive a contradiction for each case.

(a) By Corollary 5 we may assume that $n_4 \leq 2$. By Lemma 9, $f(u_6 u_{7+n_4}) = f(v_7 v_8)$, which contradicts the fact that the edges $u_6 u_{7+n_4}$ and $v_7 v_8$ are within distance two apart.

(b) Since $n_4 = 1$, by Lemma 9, $f(u_6 u_8) = f(v_7 v_8)$, which contradicts the fact that the edges $u_6 u_8$ and $v_7 v_8$ are distance two apart.

(c) By Corollary 5 we may assume that $n_5 \leq 2$. By Lemma 9 and Corollary 5, $f(u_6 u_{10}) = f(v_7 v_8) = f(u_9 u_{10+n_5})$. For the case $n_5 = 1$, this is a contradiction as $u_6 u_{10}$ and $u_9 v_{11}$ are of distance two apart. For the case $n_5 = 2$, by Lemma 3 (i), $f(u_6 u_{10}) = f(v_{11} v_{12})$, and so $f(v_{11} v_{12}) = f(u_9 v_{12})$, a contradiction.

The proofs for (d), (e), and (f) are similar. We leave the details to the reader. □

An immediate consequence of Theorem 10 is that for every integer $k \geq 7$, there exists $G \in \mathcal{G}_k$ with $\chi'_s(G) = 7$. This gives infinite counter examples to Conjecture 3.

4 Special Families

We apply the results and methods established in the previous sections to completely determine the value of $\chi'_s(G)$ for several families of graphs G in $\mathcal{G}_k$.

Corollary 11. *For $m \geqslant 1$, we have*

$$\chi'_s(G_{1\star m}) = \begin{cases} 9, & m = 2; \\ 7, & m = 4, 6; \\ 6, & \textit{otherwise}. \end{cases}$$

Proof. For $m = 2$, as $G_{1,1} = G_2$ and $\chi'_s(G_2) = 9$ [18], so the result holds. For $m = 4$, $G_{1\star 4} = G_{2,2}$ so $\chi'_s(G_{1\star 4}) = 7$.

Because $G_3 = G_{1\star 3}$ and $\chi'_s(G_3) = 6$, so $\chi'_s(G_{1\star 3}) = 6$. By Corollary 8 (letting $n_1 = n_2 = \cdots = n_{r-1} = 1$) and Figure 7 the result holds for $m = 5$ and $m \geqslant 7$.

It remains to show that $\chi'_s(G_{1\star 6}) > 6$. Assume to the contrary $\chi'_s(G_{1\star 6}) = 6$. As $G_{1\star 6} = G_{2,1,1,2}$, we may let f be a strong 6-edge-coloring for $G_{2,1,1,2}$. Without loss of generality, assume $f(v_0u_1) = 1$, $f(v_0v_1) = 2$, $f(v_0u_3) = 3$, $f(u_1u_2) = 4$, $f(v_1u_1) = 5$, and $f(v_1v_2) = 6$. Then $f(v_2u_2) = 3$, $f(v_2v_3) = 1$, and $f(u_2u_4) = 2$. These imply that $\{f(v_3v_4) = f(v_3u_3)\} = \{4, 5\}$, so $f(v_4u_4) = f(v_0u_1) = 6$. By Lemma 3, it must be $f(v_6v_7) = 6$, which is a contradiction as v_6v_7 and v_4u_4 are distance two apart. □

Corollary 12. *For $m \geqslant 2$, we have*

$$\chi'_s(G_{2\star m}) = \begin{cases} 7, & m = 2; \\ 6, & \textit{otherwise}. \end{cases}$$

Proof. At the beginning of Section 3, we have learned that $\chi'_s(G_{2,2}) = 7$. Figure 6 shows a strong 6-edge coloring f for $G_{3\star 2}$. In the following we define a recursive strong 6-edge coloring for $G_{2\star m}$, $m \geq 3$.

Initially, let the coloring in Figure 6 be f_2. Suppose we have a strong 6-edge coloring f_m for $G_{2\star m}$. Extend f_m to a strong 6-coloring f_{m+1} for $G_{2\star(m+1)}$ by:

$f_{m+1}(ww') = f_m(ww')$ if $ww' \in E(G_{2\star m})$;

$f_{m+1}(v_{2m+1}v_{2m+2}) = f_m(u_{2m-1}u_{2m})$;

$f_{m+1}(u_{2m-2}u_{2m+1}) = f_{m+1}(v_{2m+2}v_{2m+3}) = f_m(v_{2m+1}u_{2m-2})$;

$f_{m+1}(u_{2m}u_{2m+3}) = f_{m+1}(v_{2m+1}u_{2m+1}) = f_m(v_{2m+1}u_{2m})$;

$f_{m+1}(v_{2m+2}u_{2m+2}) = f_m(v_{2m-1}v_{2m})$;

$f_{m+1}(u_{2m+1}u_{2m+2}) = f_m(v_{2m}u_{2m})$; and

$f_{m+1}(v_{2m+3}u_{2m+2}) = f_m(v_{2m}v_{2m+1})$.

It is easy to check that the above is a strong 6-edge coloring for $G_{2\star(m+1)}$. We shall leave the details to the reader. □

Corollary 13. *For $m \geqslant 1$, we have*

$$\chi'_s(G_{3\star m}) = \begin{cases} 7, & m = 2,4,6; \\ 6, & \textit{otherwise.} \end{cases}$$

Proof. We first consider $m \neq 2,4,5$. Since $\chi'_s(G_3) = \chi'_s(G_5) = 6$, by Observations 1 and 2 we have $\chi'_s(G_{1,3,1}) = 6$. By Corollary 5, we get $\chi'_s(G_{3,3,3}) = 6$. Hence, the result holds for $m = 1,3$.

Assume $m \geq 6$. If $\chi'_s(G_{2,3\star(m-4),2}) = 6$, then by Corollary 5, $\chi'_s(G_{4,3\star(m-4),4}) = \chi'_s(G_{1,3\star(m-2),1}) = \chi'_s(G_{3\star m}) = 6$. Hence, it is enough to find a strong 6-edge-coloring f for $G_{2,3\star(m-4),2}$.

In the following we let $f(v_0u_1) = 1$, $f(v_0v_1) = 2$, $f(v_0u_3) = 3$, $f(u_1u_2) = 4$, $f(v_1u_1) = 5$, and $f(v_1v_2) = 6$. Consequently, by Lemma 3, $f(v_2v_3) = f(u_4u_5) = 1$, $f(u_2u_6) = f(v_7v_8) = 2$, and $f(u_2v_2) = f(v_4v_5) = 3$. Since $f(v_5v_6), f(u_2u_6) \neq 4$, so the color 3 has to be used in the 4-cycle $(u_6u_7v_7v_6)$, it must be the case that $f(v_7u_7) = 3$.

Assume m is even. Let $m - 4 = 2k$. Define $f(v_4u_4) = 4$, $f(u_3u_4) = 6$, and the remaining by the following recursive process for $1 \leq t \leq 2k$:

$$f(v_{3t}v_{3t+1}) = \begin{cases} f(v_{3t-2}u_{3t-2}) & \text{if } t \text{ is even;} \\ f(v_{3t-3}v_{3t-2}) & \text{if } t \text{ is odd.} \end{cases}$$

$$f(v_{3t}u_{3t}) = \begin{cases} f(u_{3t-2}u_{3t-1}) & \text{if } t \text{ is even;} \\ f(v_{3t-2}u_{3t-2}) & \text{if } t \text{ is odd.} \end{cases}$$

$$f(u_{3t}u_{3t+1}) = \begin{cases} f(v_{3t-1}u_{3t-1}) & \text{if } t \text{ is even;} \\ f(u_{3t-2}u_{3t-1}) & \text{if } t \text{ is odd, } t \geq 3. \end{cases}$$

$$f(v_{3t+1}u_{3t+1}) = \begin{cases} f(v_{3t-2}v_{3t-1}) & \text{if } t \text{ is even;} \\ f(v_{3t-1}u_{3t-1}) & \text{if } t \text{ is odd, } t \geq 3. \end{cases}$$

By Lemma 3, the colors for the remaining edges are fixed. It is not hard to see that f is a strong 6-edge-coloring for $G_{2,3\star(2k),2}$. See Figure 9 for an example.

Assume m is odd. Let $m-4 = 2k+1$. Let $f(v_3v_4) = 4$, $f(v_3u_3) = 5$, $f(u_3u_4) = 6$, and $f(u_4v_4) = 2$. For $2 \leq t \leq 2k$, define f by the following recursive process:

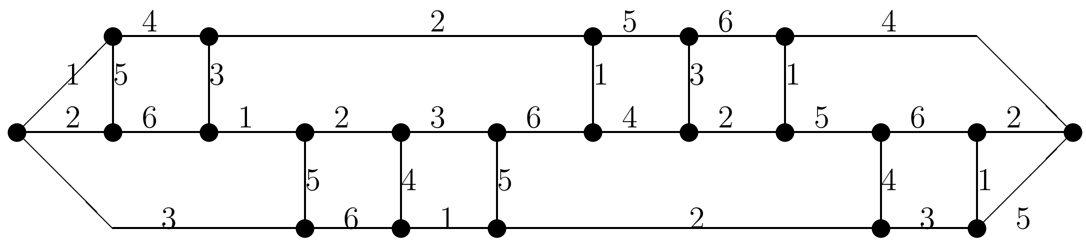

Figure 9: A strong 6-edge-coloring for $G_{2,3,3,2}$.

$$f(v_{3t}v_{3t+1}) = \begin{cases} f(v_{3t-3}v_{3t-2}) & \text{if } t \text{ is even;} \\ f(u_{3t-2}u_{3t-1}) & \text{if } t \text{ is odd.} \end{cases}$$

$$f(v_{3t}u_{3t}) = \begin{cases} f(u_{3t-2}u_{3t-1}) & \text{if } t \text{ is even;} \\ f(v_{3t-2}u_{3t-2}) & \text{if } t \text{ is odd.} \end{cases}$$

$$f(u_{3t}u_{3t+1}) = \begin{cases} f(v_{3t-1}u_{3t-1}) & \text{if } t \text{ is even;} \\ f(v_{3t-2}v_{3t-1}) & \text{if } t \text{ is odd.} \end{cases}$$

$$f(v_{3t+1}u_{3t+1}) = \begin{cases} f(u_{3t-2}v_{3t-2}) & \text{if } t \text{ is even, } t \neq 2; \\ f(u_{3t-1}v_{3t-1}) & \text{if } t \text{ is odd.} \end{cases}$$

Note, for $t = 2$ in the last case above, $f(v_7u_7) = 3$ is fixed as discussed at the beginning of the proof.

For $t = 2k+1$, let $f(v_{6k+3}v_{6k+4}) = f(u_{6k+1}v_{6k+1})$, $f(v_{6k+3}u_{6k+3}) = f(u_{6k+1}u_{6k+2})$, $f(u_{6k+3}u_{6k+4}) = f(v_{6k+1}v_{6k+2})$, and $f(v_{6k+4}u_{6k+4}) = f(u_{6k+2}v_{6k+2})$.

Again, by Lemma 3, the colors for the remaining edges are fixed. It is not hard to see that f is a strong 6-edge-coloring for $G_{2,3\star(2k+1),2}$. See Figure 10 for an example.

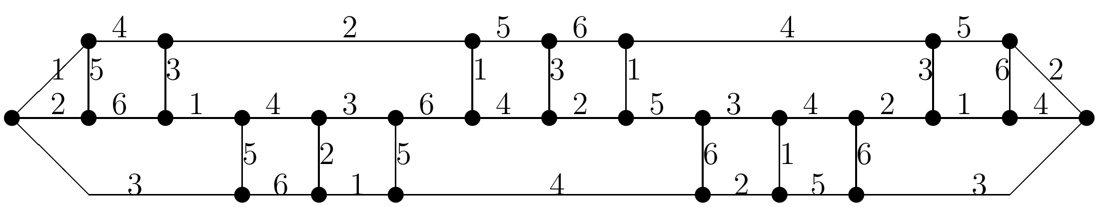

Figure 10: A strong 6-edge-coloring for $G_{2,3,3,3,2}$.

Now consider $m = 2$. Because $\chi'_s(G_{3,1}) = \chi'_s(G_4) > 6$, Corollary 5 implies that $\chi'_s(G_{3,3}) > 6$, so $\chi'_s(G_{3,3}) = 7$.

For $m = 4$, since $\chi'_s(G_{2,2}) > 6$, by Corollary 5, we get $\chi'_s(G_{4,4}) = \chi'_s(G_{1,3,3,1}) > 6$. Use Corollary 5 twice again, we obtain $\chi'_s(G_{3,3,3,3}) > 6$, so $\chi'_s(G_{3,3,3,3}) = 7$.

For $m = 5$, by Theorem 10 (a), $\chi'_s(G_{2,3,1,1}) = \chi'_s(G_{2,3,2}) = 7$. This implies, by Corollary 5, $\chi'_s(G_{4,3,4}) = \chi'_s(G_{1,3,3,3,1}) = \chi'_s(G_{3\star 5}) = 7$. □

Acknowledgments. The authors thank the referees for their prompt reports with many constructive suggestions.

References

[1] L. D. Andersen, The strong chromatic index of a cubic graph is at most 10, Discrete Math. 108 (1992) 231 – 252.

[2] R. A. Brualdi and J. Q. Massey, Incidence and strong edge colorings of graphs, Discrete Math. 122 (1993) 51 – 58.

[3] K. Cameron, Induced matchings, Discrete Appl. Math. 24 (1989) 97 – 102.

[4] D. Cranston, Strong edge-coloring graphs with maximum degree 4 using 22 colors, Discrerte Math. 306 (2006) 2772 – 2778.

[5] P. Erdős, Problems and results in combinatorial analysis and graph theory, Discrete Math. 72 (1988) 81 – 92.

[6] P. Erdős and J. Nešetřil, [Problem], in: G. Halász and V. T. Sós (eds.), Irregularities of Partitions, Springer, Berlin, 1989, 162 –163.

[7] R. J. Faudree, R. H. Schelp, A. Gyárfás and Zs. Tuza, The strong chromatic index of graphs, Ars Combin. 29B (1990) 205 – 211.

[8] J. L. Fouquet and J. Jolivet, Strong edge-coloring of graphs and applications to multi-k-gons, Ars Combin. 16A (1983) 141 – 150.

[9] J. L. Fouquet and J. Jolivet, Strong edge-coloring of cubic planar graphs, Progress in Graph Theory (Waterloo 1982), 1984, 247 – 264.

[10] M. C. Golumbic and M. Lewenstein, New results on induced matchings, Discrete Appl. Math. 101 (2000) 157 – 165.

[11] P. Horák, The strong chromatic index of graphs with maximun degree four, Contemp. Methods Graph Theory, 1990, 399 – 403.

[12] P. Horák, H. Qing and W. T. Trotter, Induced matchings in cubic graphs, J. Graph Theory 17 (1993) 151 – 160.

[13] H.-H. Lai, K.-W. Lih and P.-Y. Tsai, The strong chromatic index of Halin graphs, Discrete Math. (2011), doi:10.1016/j.disc.2011.09.016.

[14] K.-W. Lih and D. D.-F. Liu, On the strong chromatic index of cubic Halin graphs, Appl. Math. Lett. (2011), doi:10.1016/j.aml.2011.10.046.

[15] M. Mahdian, On the computational complexity of strong edge coloring, Discrete Appl. Math. 118 (2002) 239 – 248.

[16] M. Maydanskiy, The incidence coloring cofor graphs of maximum degree 3, Discrete Math. 292 (2005) 131 – 141.

[17] M. R. Salavatipour, A polynomial time algorithm for strong edge coloring of partial k-trees, Discrete Appl. Math. 143 (2004) 285 – 291.

[18] W. C. Shiu, P. C. B. Lam and W. K. Tam, On strong chromatic index of Halin graphs, J. Combin. Math. Combin. Comput. 57 (2006) 211 – 222.

[19] W. C. Shiu and W. K. Tam, The strong chromatic index of complete cubic Halin graphs, Appl. Math. Lett. 22 (2009) 754 – 758.

[20] J. Wu and W. Lin, The strong chromatic index of a class of graphs, Discrete Math. 308 (2008) 6254 – 6262.