Math 2130 - Test 2

Name: Solutions

Directions: Show all of your work to get credit. No calculators. Good luck!

Score	
1	
2	
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1. [5 points] Consider the function

$$f(x, y) = y^2 - xy + 2x + y + 1$$

Find the critical points of f and determine if they are local minimums, local maximums, or saddle points. Recall that $D = f_{xx}f_{yy} - f_{xy}^2$.

$$\left.\begin{aligned} 0 &= f_x = -y+2 \\ 0 &= f_y = 2y-x+1 \end{aligned}\right\} \quad \left.\begin{aligned} 0 &= -y+2 \\ 0 &= 2y-x+1 \end{aligned}\right\} \quad \begin{aligned} y &= 2 \\ 0 &= 2(2)-x+1 \\ x &= 5 \end{aligned}$$

Critical points: $(x,y) = (5,2)$

$f_{xx} = 0$

$f_{yy} = 2$

$f_{xy} = -1$

$D(5,2) = (0)(2) - [-1]^2 = -1 < 0$

So, $(5,2)$ is a saddle point.

2. [5 points] Find the maximum and minimum values of

$$f(x, y) = x^2 + y^2 - 4y + 4$$

subject to the constraint $x^2 + y^2 = 1$.

Use Lagrange multipliers.

$\nabla f = \langle 2x, 2y-4 \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$\boxed{\begin{array}{l} \nabla f = \lambda \nabla g \\ g = k \end{array}} \rightarrow \boxed{\begin{array}{ll} 2x = \lambda 2x & (1) \\ 2y-4 = 2\lambda y & (2) \\ x^2+y^2 = 1 & (3) \end{array}}$

Consider (1) $2x = \lambda 2x$, which is $2x(1-\lambda) = 0$.
So either $\lambda = 1$ or $x = 0$.

case 1: $\lambda = 1$. Plug this into (2) to get $2y - 4 = 2y$.
This gives $-4 = 0$. So there is no solution with $\lambda = 1$.

case 2: $x = 0$: Plug this into (3) to get $y^2 = 1$.
So, $y = \pm 1$. [Note: If $y = 1$, then $\lambda = -1$ in (2). If $y = -1$, then $\lambda = +3$ in (2).]

Now we plug $(x,y) = (0,1), (0,-1)$ into f

$f(0,1) = 0^2 + 1^2 - 4(1) + 4 = 1$

$f(0,-1) = 0^2 + (-1)^2 - 4(-1) + 4 = 9$

1 is the min
9 is the max

3. [5 points] Evaluate the following double integral.

$$\int_0^{ln(2)} \int_0^1 ye^{xy}\, dx\, dy$$

$$\int_0^{\ln(2)} \int_0^1 ye^{xy}\, dx\, dy = \int_0^{\ln(2)} \left(y \cdot \frac{1}{y} e^{xy} \Big|_{x=0}^{1} \right) dy$$

$$= \int_0^{\ln(2)} (e^{1\cdot y} - e^{0\cdot y})\, dy = \int_0^{\ln(2)} (e^{y} - 1)\, dy$$

$$= (e^{y} - y)\Big|_{y=0}^{\ln(2)} = (e^{\ln(2)} - \ln(2)) - (e^{0} - 0)$$

$$= 2 - \ln(2) - 1 = \boxed{1 - \ln(2)}$$

Note: You can also treat this with a u-sub.

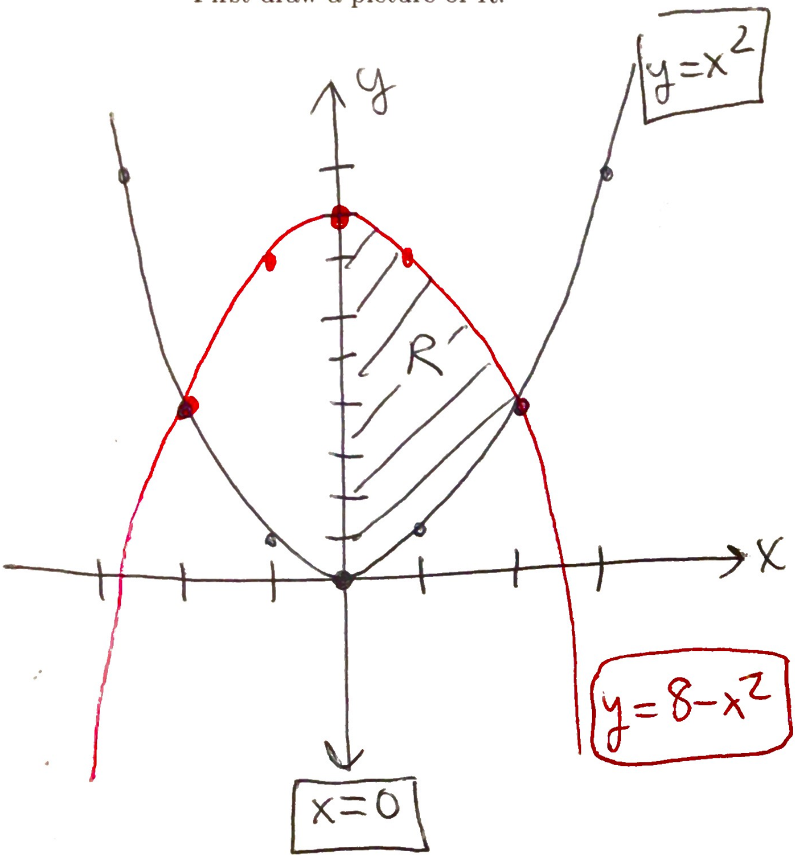
$$\int ye^{xy}\, dx \overset{\boxed{\begin{array}{l} u = xy \\ du = y\, d\cancel{x} \\ \frac{du}{y} = d\cancel{x} \end{array}}}{=} \int y \cdot e^{u} \cdot \frac{du}{y} = \int e^{u} du = e^{u} = e^{xy}$$

4. [5 points] Compute the integral

$$\iint_R x\, dA$$

where R is the region in the first quadrant that is bounded by $x = 0$ and $y = x^2$ and $y = 8 - x^2$.

First draw a picture of R.



R is parameterized by

$0 \leq x \leq 2$

$x^2 \leq y \leq 8-x^2$

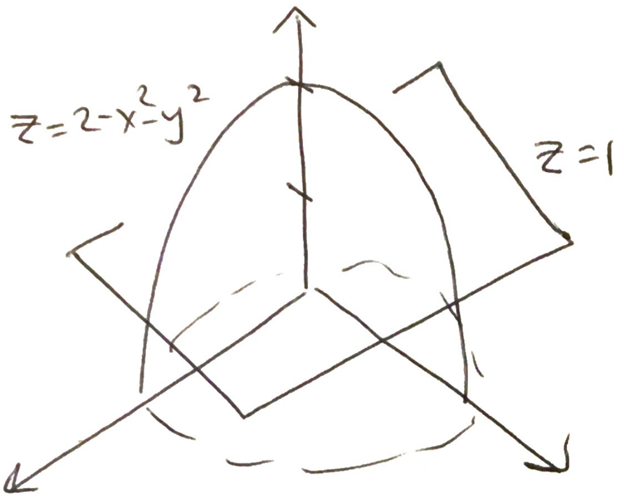
$$\iint_R x\, dA = \int_0^2 \int_{x^2}^{8-x^2} x\, dy\, dx = \int_0^2 xy \Big|_{y=x^2}^{8-x^2} dx$$

$$= \int_0^2 x(8-x^2) - x(x^2)\, dx = \int_0^2 (8x - x^3 - x^3)\, dx$$

$$= \int_0^2 (-2x^3 + 8x)\, dx = \left(-2\frac{x^4}{4} + 8\frac{x^2}{2}\right)\Big|_0^2 = \left(-\frac{x^4}{2} + 4x^2\right)\Big|_0^2$$

$$= -\frac{(2)^4}{2} + 4(2)^2 = -8 + 16 = 8$$

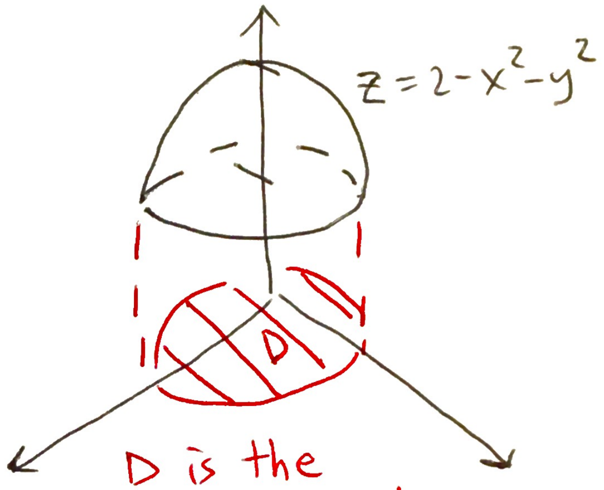
5. [5 points] Find the volume of the solid that is bounded by the paraboloid $z = 2 - x^2 - y^2$ and the plane $z = 1$.



These surfaces meet at

$$2 - x^2 - y^2 = z = 1$$

$$x^2 + y^2 = 1$$



D is the filled in circle of radius 1 ie $x^2 + y^2 \leq 1$

D is parameterized by
$0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$

Volume is

$$\iint_D \left[(2 - x^2 - y^2) - 1\right] dA$$

$$= \iint_D \left[1 - (x^2 + y^2)\right] dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r - r^3)\, dr \, d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4}\right) d\theta = \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{1}{4}\theta \Big|_0^{2\pi} = \frac{1}{4}(2\pi) = \frac{\pi}{2}$$