Math 2150-01
1/29/25

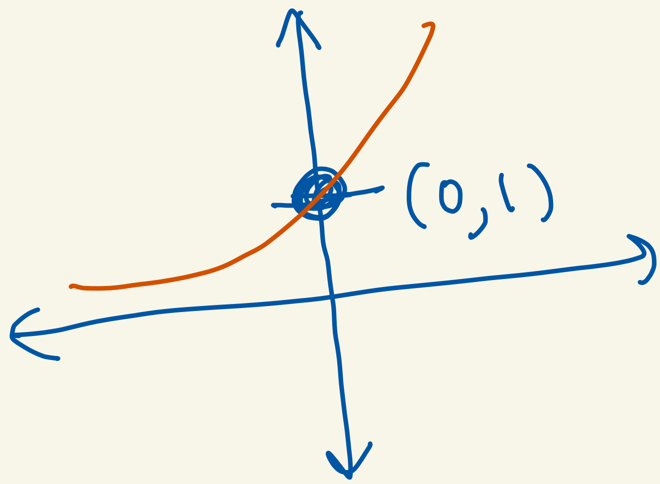
(Topic 1 continued...)

Ex: Let's find a solution to the initial-value problem

$$\boxed{\begin{aligned} y' &= y^2 \\ y(0) &= 1 \end{aligned}}$$

nonlinear ODE

condition on solution



Consider $f(x) = \dfrac{1}{1-x}$

Then: $f(x) = (1-x)^{-1}$

$$f'(x) = -(1-x)^{-2} \cdot (-1)$$

$$= (1-x)^{-2} = \frac{1}{(1-x)^2}$$

Then,

$$\underbrace{f'(x)}_{y'} = \frac{1}{(1-x)^2} = \left[\frac{1}{1-x}\right]^2 = \underbrace{[f(x)]^2}_{y^2}$$

So,

$f(x) = \frac{1}{1-x}$ satisfies $y' = y^2$,

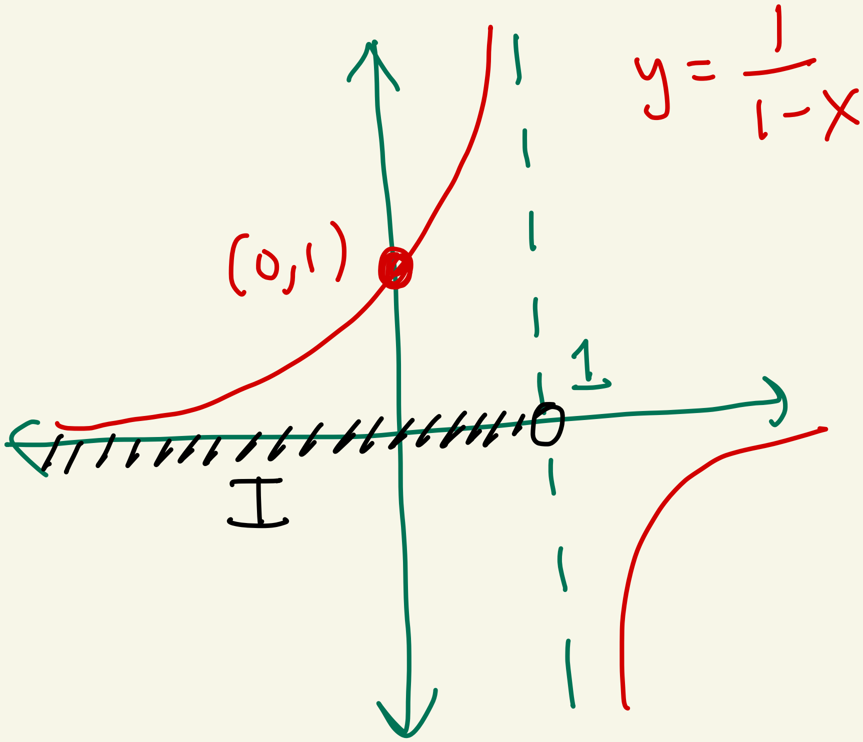
Also,

$$f(0) = \frac{1}{1-0} = 1$$

checking! $y(0)=1$

So, f satisfies the problem

You could say f solves the problem on $I = (-\infty, 1)$



HW 1
2(d,e)

2(d) Given any constants c_1 and c_2 show that

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

satisfies

$$y'' - 4y = 0$$

on $I = (-\infty, \infty)$

<u>Ex:</u> $c_1 = 5, c_2 = -3$

$$f(x) = 5e^{2x} - 3e^{-2x}$$

We get:

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

$$f''(x) = 4c_1 e^{2x} + 4c_2 e^{-2x}$$

these exist for all x that is on $I = (-\infty, \infty)$

Plug in $y'' = f''$ and $y = f$ to get:

$$y'' - 4y = (4c_1 e^{2x} + 4c_2 e^{-2x}) - 4(c_1 e^{2x} + c_2 e^{-2x})$$

$$= 0$$

So, f satisfies $y'' - 4y = 0$ on $I = (-\infty, \infty)$.

$\boxed{\text{END } 2(d)}$

$\boxed{2(e)}$ Find c_1, c_2 where

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

solves the initial-value problem

$$\boxed{\begin{aligned} & y'' - 4y = 0 \\ & y'(0) = 0 \\ & y(0) = 1 \end{aligned}}$$

ODE

extra conditions on the solution

We already know from 2(d) that $f(x) = c_1 e^{2x} + c_2 e^{-2x}$ solves $y'' - 4y = 0$.

Let's make it solve the extra conditions,

We have

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

We need:

$$\boxed{\begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} c_1 e^{2(0)} + c_2 e^{-2(0)} &= 1 \\ 2c_1 e^{2(0)} - 2c_2 e^{-2(0)} &= 0 \end{aligned}}$$

$$\xrightarrow{\boxed{e^0 = 1}}$$

$$\boxed{\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 - 2c_2 &= 0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} c_1 + c_2 &= 1 \quad (1) \\ c_1 - c_2 &= 0 \quad (2) \end{aligned}}$$

(2) gives $c_1 = c_2$.

Plug $c_1 = c_2$ into (1) to get

$c_2 + c_2 = 1$. So, $\boxed{c_2 = \frac{1}{2}}$.

Plug back into $c_1 = c_2$ to get $\boxed{c_1 = \frac{1}{2}}$ also.

So,

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$
$$= \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}$$

is the solution.

END OF 2(e)

SKIPPING TOPIC 2

Topic 3 - First order linear ODEs

We will give a method to solve

$$y' + a(x)y = b(x)$$

on any interval I where $a(x), b(x)$ are continuous.

Ex: $b(x) = x$

$$y' + \underbrace{2x}_{a(x)=2x}y = \overbrace{x}^{b(x)}$$

$a(x) = 2x$

$I = (-\infty, \infty)$

Since $a(x)$ is continuous we can find an antiderivative

$$A(x) = \int a(x)\,dx$$

So, $A'(x) = a(x)$

Ex

$$A(x) = \int 2x\,dx = x^2$$

Multiply $y' + a(x)y = b(x)$ by $e^{A(x)}$ to get:

$$\underbrace{e^{A(x)}y' + e^{A(x)}a(x)y}_{\left(e^{A(x)}\cdot y\right)'} = b(x)e^{A(x)}$$

We get

$$\left(e^{A(x)}\cdot y\right)' = b(x)e^{A(x)}$$

Integrate both sides with respect to x to get:

$$e^{A(x)}y = \int b(x)e^{A(x)}dx$$

Ex

$$y' + 2xy = x$$

$$e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}$$

Ex

$$\left(e^{x^2}\cdot y\right)' = xe^{x^2}$$

Ex

$$e^{x^2}y = \frac{1}{2}e^{x^2} + C$$

Solve for y:

$$y = e^{-A(x)} \int b(x) e^{A(x)} dx$$

Since you can reverse the above steps this is the only solution to the ODE.

$$y = \frac{1}{2} \underbrace{e^{-x^2} e^{x^2}}_{e^{-x^2+x^2} = e^0 = 1} + Ce^{-x^2}$$

$$y = \frac{1}{2} + Ce^{-x^2}$$