Math 2150 - 01
2/17/25

Ex: Last time we looked at

$$2xy + (x^2-1)y' = 0$$

To solve it we need to find $f(x,y)$ where

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2xy \quad (1) \\ \frac{\partial f}{\partial y} &= x^2 - 1 \quad (2) \end{aligned}}$$

Let's see how to solve this with our second method.

Integrate (1) with respect to x:

$$f(x,y) = x^2 y + \underbrace{C(y)}_{\text{constant with respect to } x} \quad (3)$$

Integrate (2) with respect to y:

$$f(x,y) = x^2y - y + \underbrace{D(x)}_{\text{constant with respect to } y} \quad (4)$$

Set (3) equal to (4):

$$\cancel{x^2y} + C(y) = \cancel{x^2y} - y + D(x)$$

Simplify:

$$C(y) = -y + \underbrace{D(x)}_{0}$$

Set $C(y) = -y$ and $D(x) = 0$.

Plug either into (3) or (4) to find f.

Let's plug $C(y) = -y$ into (3) to get

$$f(x,y) = x^2y + C(y)$$
$$= x^2y - y$$

This is the answer we got last time

$$\boxed{\text{Topic 6 - Theory of second order linear ODEs}}$$

So far we've been solving first order equations.

Now we switch to second order. We will look at these:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

(2nd order linear)

To do this we need some preliminaries.

Def: Let I be an interval.
Let f_1 and f_2 be defined on I.
We say that f_1 and f_2 are linearly dependent if either

① $f_1(x) = c f_2(x)$ for all x in I

or

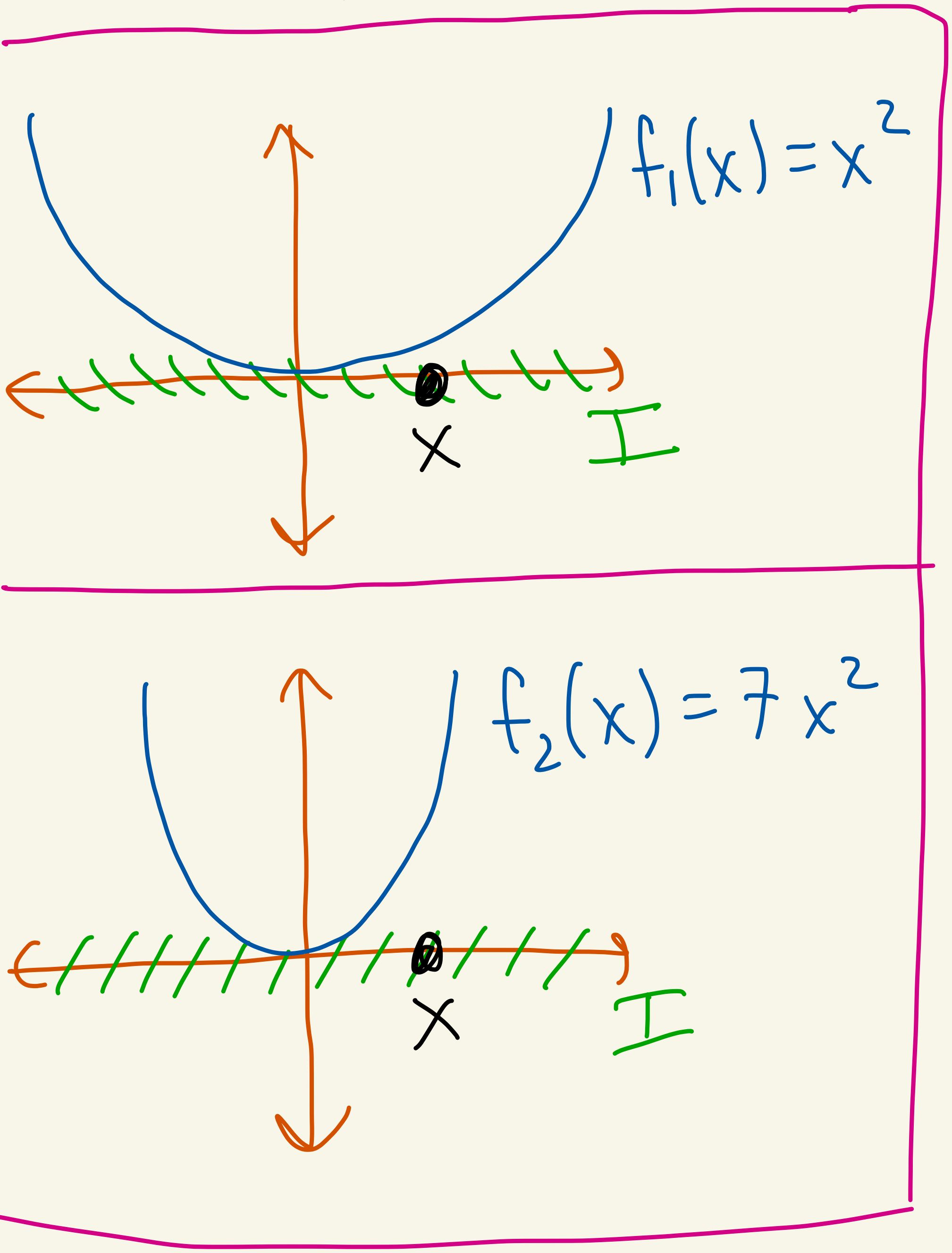
② $f_2(x) = c f_1(x)$ for all x in I

where c is a constant.

If no such c exists then f_1, f_2 are called linearly independent.

Ex: Let $I = (-\infty, \infty)$.

Let $f_1(x) = x^2$ and $f_2(x) = 7x^2$.



f_1 and f_2 are linearly dependent because for example

$f_1(x) = \frac{1}{7} f_2(x)$

for all x in I.

Or you could say

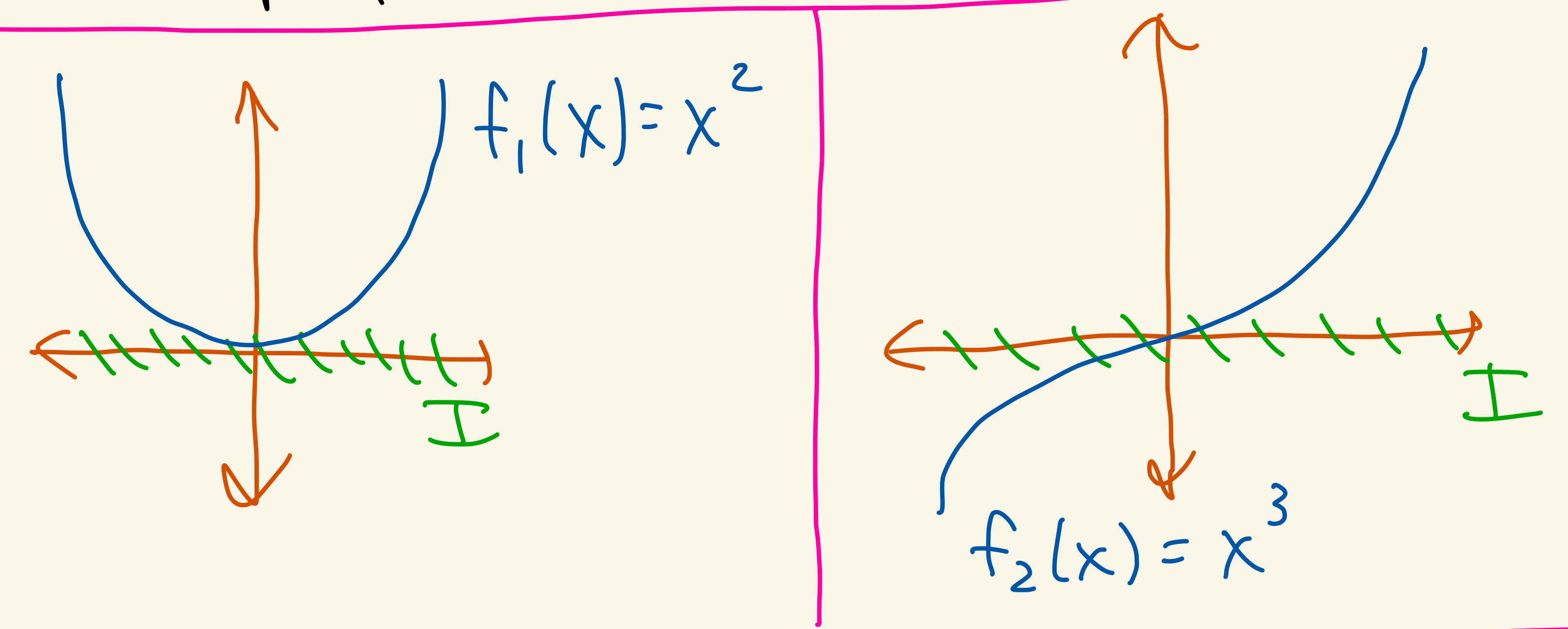
$f_2(x) = 7 f_1(x)$

for all x in I

Ex: Let $I=(-\infty,\infty)$.

Let $f_1(x)=x^2$ and $f_2(x)=x^3$.

These functions are linearly independent. Why?



Suppose $f_1(x)=c f_2(x)$
for all x in I.

Then $x^2=cx^3$ for all x.

Plug in $x=1$ to get $1=c$.

Plug in $x=2$ to get $\frac{1}{2}=c$

This is nonsense!

Similarly you cant have $f_2(x) = c f_1(x)$.

They must be linearly independent!!!

We will learn another way to check this based on the Wronskian.

Josef Wronski (1778-1853)

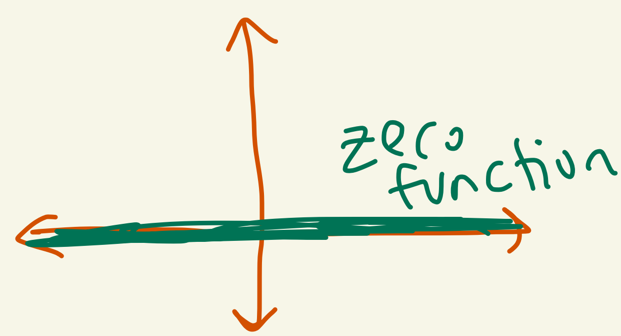
Theorem: Let I be an interval. Let f_1, f_2 be differentiable on I. If the Wronskian

$$\underbrace{W(f_1,f_2)}_{\text{notation}} = \underbrace{\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}}_{\text{notation for determinant}} = \underbrace{f_1 f_2' - f_2 f_1'}$$

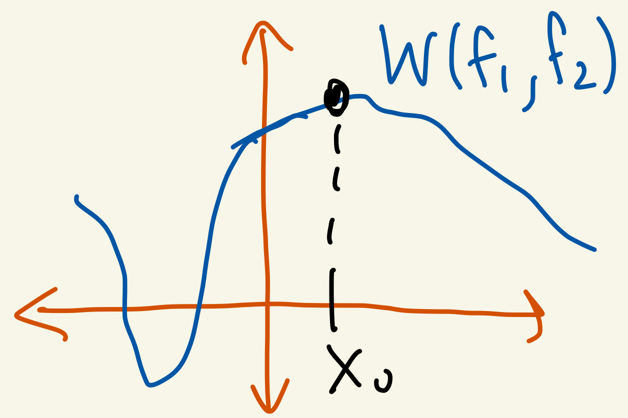
picture

$$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

is not the zero function, then f_1 and f_2 are linearly independent.



That is, if there exists an x_0 in I with $W(f_1,f_2)(x_0) \neq 0$ then f_1, f_2 are linearly independent



Ex: Let $I=(-\infty,\infty)$ and $f_1(x)=e^{2x}$, $f_2(x)=e^{5x}$.

Let's show these functions are linearly independent.

$$W(f_1,f_2)=\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$=\begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix}$$

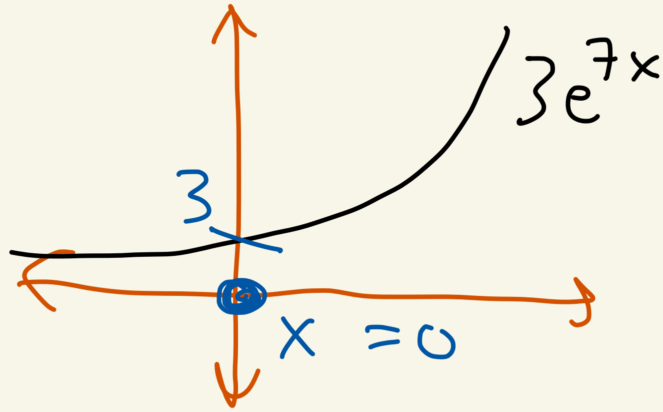
$$=(e^{2x})(5e^{5x})-(2e^{2x})(e^{5x})$$

$$=5e^{7x}-2e^{7x}$$

$$=3e^{7x}$$

is this the zero function?

Plug in $x = 0$
to get $3e^{7(0)} = 3 \neq 0$



Since the Wronskian is not the zero function, f_1 and f_2 are linearly independent.

<u>Theorem:</u> Let I be an interval.
Let $a_2(x)$, $a_1(x)$, $a_0(x)$ be continuous on I and $a_2(x) \neq 0$ for all x in I.

Consider the homogeneous equation

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = 0} \quad (*)$$

homogeneous when this is 0

If f_1 and f_2 are two linearly

independent solutions to $(*)$, then every solution is of the form

$$y_h = c_1 f_1(x) + c_2 f_2(x)$$

h for homogeneous

where c_1, c_2 are constants.