2150-01
2/26/25

Ex: Solve

$$y'' - 4y' + 13y = 0$$

The characteristic equation is

$$r^2 - 4r + 13 = 0$$

The roots are

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm \sqrt{36}\underbrace{\sqrt{-1}}_{}}{2}$$

i

$$= \frac{4 \pm 6i}{2} = \underbrace{2 \pm 3i}$$

$2+3i,\ 2-3i$

$\boxed{\text{RECALL:}}$

Case 3 formula:

roots: $\alpha \pm \beta i$

Solution: $y_h = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$

In our example, $\alpha \pm \beta i = 2 \pm 3i$

So, $\alpha = 2$, $\beta = 3$.

Summary! The general solution to

$$y'' - 4y' + 13y = 0$$

is

$$y_h = c_1 e^{2x}\cos(3x) + c_2 e^{2x}\sin(3x)$$

HW 7
2(a)

Solve:

$$4y'' - y' = 0$$
$$y(0) = 0$$
$$y'(0) = 0$$

First we solve

$$4y'' - y' = 0$$

The characteristic equation is

$$4r^2 - r = 0$$

Just factor

$$\underbrace{r}_{r=0}\underbrace{(4r-1)}_{4r-1=0} = 0$$

$r = 1/4$

So we get two distinct real roots $r_1 = 0$, $r_2 = 1/4$.

So, the general solution to $4y'' - y' = 0$ is:

$$y_h = c_1 e^{0x} + c_2 e^{\frac{1}{4}x}$$

$$= c_1 + c_2 e^{x/4}$$

case 1 from mon. theorem

$e^{0x} = e^0 = 1$

Now we make the solution also satisfy $y(0) = 0$, $y'(0) = 0$.

We have

$$y_h = c_1 + c_2 e^{x/4}$$

$$y_h' = \frac{1}{4} c_2 e^{x/4}$$

Need to solve:

$$\boxed{\begin{aligned} y_h(0) &= 0 \\ y_h'(0) &= 0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} c_1 + c_2 e^{0/4} &= 0 \\ \frac{1}{4} c_2 e^{0/4} &= 0 \end{aligned}}$$

$$e^{0/4} = e^0 = 1$$

$$\boxed{\begin{aligned} c_1 + c_2 &= 0 \quad (1) \\ \frac{1}{4} c_2 &= 0 \quad (2) \end{aligned}}$$

(2) says $c_2 = 0$

Plug $c_2 = 0$ into (1) to get $c_1 + 0 = 0$

So, $c_1 = 0$.

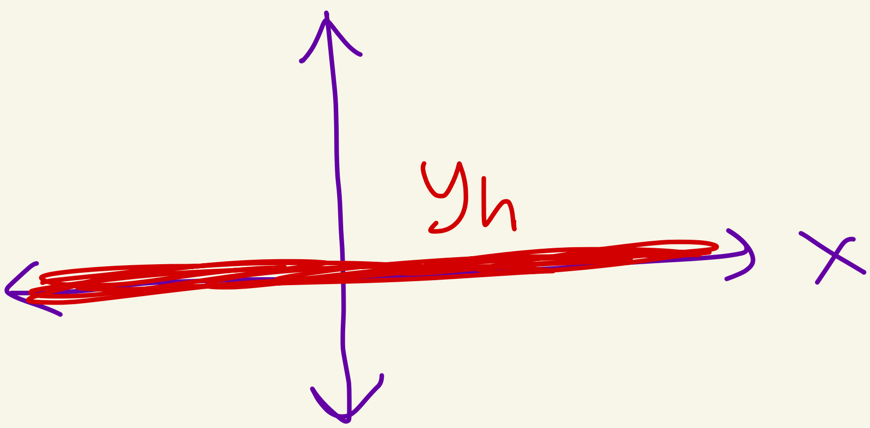
Thus,

$$y_h = c_1 + c_2 e^{x/4}$$

$$= 0 + 0 e^{x/4}$$

$$= 0$$

$\boxed{\text{So, } y_h = 0 \text{ is the solution to } 4y'' - y' = 0,\ y(0) = 0,\ y'(0) = 0}$



Let's see why the case 1 formula works.

Consider

$$a_2 y'' + a_1 y' + a_0 y = 0.$$

Let's guess a solution.

Consider $y = e^{rx}$ where r is a real number.

Then,

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

Plug this into the left side of the differential equation to get:

$$a_2 y'' + a_1 y' + a_0 y$$

$$= a_2(r^2 e^{rx}) + a_1(re^{rx}) + a_0(e^{rx})$$

$$= \underbrace{e^{rx}}_{\text{never zero}} \underbrace{\left[a_2 r^2 + a_1 r + a_0\right]}_{\text{characteristic equation}}$$

Thus, $y = e^{rx}$ satisfies $a_2 y'' + a_1 y' + a_0 y = 0$ when $a_2 r^2 + a_1 r + a_0 = 0$.

Now lets look at case 1 of Mondays theorem.

Let r_1, r_2 be distinct real roots of $a_2 r^2 + a_1 r + a_0 = 0$

Then from the analysis above we know

$f_1(x) = e^{r_1 x}$ and $f_2(x) = e^{r_2 x}$

are solutions to

$$a_2 y'' + a_1 y' + a_0 y = 0.$$

Let's show f_1, f_2 are linearly independent.

We have

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

$$= \left(e^{r_1 x}\right)\left(r_2 e^{r_2 x}\right) - \left(r_1 e^{r_1 x}\right)\left(e^{r_2 x}\right)$$

$$= r_2 e^{(r_1 + r_2)x} - r_1 e^{(r_1 + r_2)x}$$

$$= \underbrace{(r_2 - r_1)}_{\substack{r_2 - r_1 \neq 0 \\ \text{since} \\ r_1 \neq r_2}} \underbrace{e^{(r_1 + r_2)x}}_{\substack{\text{never} \\ 0}} \neq 0 \quad \text{for all } x$$

Thus, $f_1(x) = e^{r_1 x}$, $f_2(x) = e^{r_2 x}$ are linearly independent solutions to $a_2 y'' + a_1 y' + a_0 y = 0$

Thus, by topic 6, the general solution to $a_2 y'' + a_1 y' + a_0 y = 0$ is $y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

See notes for case 2/case 3 if you want