2150-01
3/19/25

Topic 9 - Variation of Parameters

Topic 9 is another way to find y_p.
It will work in situations where topic 8 doesn't like

$$y'' + y = \tan(x)$$

Also topic 9 will let us solve equations where the coefficients aren't all constants like:

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

DERIVATION TIME

Suppose you already have the general solution to the homogeneous equation

$$\boxed{y'' + a_1(x)y' + a_0(x)y = 0}$$

and it is

$$y_h = c_1 y_1 + c_2 y_2$$

where y_1, y_2 are linearly independent.

Using y_1 and y_2 it is possible to find a solution to

$$\boxed{y'' + a_1(x)y' + a_0(x)y = b(x)}$$

To do this set

$$y_p = v_1 \boxed{y_1} + v_2 \boxed{y_2}$$

these are homogeneous solutions

Where v_1, v_2 are unknown functions to be determined.

We will plug this in and make it work!

We need the derivatives:

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

$$= (v_1 y_1' + v_2 y_2') + \underbrace{(v_1' y_1 + v_2' y_2)}_{\text{assume } 0}$$

To simplify assume

$$\boxed{v_1' y_1 + v_2' y_2 = 0}$$

So we have

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = v_1 y_1' + v_2 y_2'$$

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Plug these into

$$y'' + a_1(x) y' + a_0(x) y = b(x)$$

to get

$$
\begin{aligned}
&\left(v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''\right) \\
&+ a_1(x)\left(v_1 y_1' + v_2 y_2'\right) \\
&+ a_0(x)\left(v_1 y_1 + v_2 y_2\right) \\
&= b(x)
\end{aligned}
$$

$$
\begin{aligned}
&y_p'' \\
&+ a_1(x) y_p' \\
&+ a_0(x) y_p \\
&= b(x)
\end{aligned}
$$

This becomes:

$$
\begin{aligned}
&v_1\big(\underbrace{y_1'' + a_1(x) y_1' + a_0(x) y_1}_{0}\big) \\
&+ v_2\big(\underbrace{y_2'' + a_1(x) y_2' + a_0(x) y_2}_{0}\big) \\
&+ \left(v_1' y_1' + v_2' y_2'\right) = b(x)
\end{aligned}
$$

The above are 0 because y_1, y_2 solve $y'' + a_1(x) y' + a_0(x) y = 0$

We are left with

$$\boxed{v_1' y_1' + v_2' y_2' = b(x)}$$

Summarizing we must solve the following for v_1', v_2':

$$\boxed{\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= b(x) \end{aligned}} \quad \begin{aligned} &1 \leftarrow \boxed{\text{simplifying assumption}} \\ &2 \leftarrow \boxed{\text{got from derivation}} \end{aligned}$$

To solve for v_2' we can calculate $y_1' * 1 - y_1 * 2$ to get:

$$(\cancel{y_1' v_1' y_1} + y_1' v_2' y_2) - (\cancel{y_1 v_1' y_1'} + y_1 v_2' y_2')$$

$$= y_1' \cdot 0 - y_1 \cdot b(x)$$

We get

$$y_1' v_2' y_2 - y_1 v_2' y_2' = -y_1 b(x)$$

So,

$$v_2' = \frac{-y_1 b(x)}{y_1' y_2 - y_1 y_2'}$$

Then

$$v_2' = \frac{y_1 b(x)}{-(y_1' y_2 - y_1 y_2')}$$

$$= \frac{y_1 b(x)}{y_1 y_2' - y_1' y_2}$$

$$= \frac{y_1 b(x)}{W(y_1, y_2)}$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Then,
$$v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)} dx$$

You can do a similar calculation $\left(y_2' * ① - y_2 * ② \right)$ to get v_1. It would give:

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx$$

Summary

Summary

Suppose you have two linearly independent solutions y_1, y_2 to

$$\boxed{y'' + a_1(x) y' + a_0(x) y = 0}$$

Then a particular solution to

$$\boxed{y'' + a_1(x) y' + a_0(x) y = b(x)}$$

is given by

$$y_p = v_1 y_1 + v_2 y_2$$

where

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx, \quad v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)} dx$$