Math 2150-01
4/21/25

HW 8-1(c)

Solve

$$\frac{1}{4}y'' + y' + y = x^2 - 2x$$

Step 1: Solve

$$\frac{1}{4}y'' + y' + y = 0$$

$$\frac{1}{4}r^2 + r + 1 = 0$$

$\times 4$

$$r^2 + 4r + 4 = 0$$

$$(r+2)(r+2) = 0$$

$$r = -2$$

$$\boxed{y_h = c_1 e^{-2x} + c_2 x e^{-2x}}$$

Step 2: Now find y_p for

$$\frac{1}{4}y'' + y' + y = x^2 - 2x$$

Guess:

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Plug these into the ODE to get

$$\underbrace{\frac{1}{4}(2A) + (2Ax + B) + (Ax^2 + Bx + c)}_{\frac{1}{4}y_p'' + y_p' + y_p} = x^2 - 2x$$

We get

$$\underbrace{A}_{A=1}x^2 + \underbrace{(2A + B)}_{2A+B=-2}x + \underbrace{\left(\frac{1}{2}A + B + c\right)}_{\frac{1}{2}A+B+C=0} = x^2 - 2x$$

So, $\boxed{A=1}$

So, $2(1)+B=-2$. Then $\boxed{B=-4}$

So, $\frac{1}{2}(1)+(-4)+C=0$. Then $\boxed{C=3.5}$

Then,

$$\boxed{y_p = Ax^2+Bx+C = x^2-4x+3.5}$$

Step 3: The general solution to

$$\frac{1}{4}y''+y'+y = x^2-2x$$

is

$$y = y_h + y_p$$

$$y = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + 3.5$$

HW 9-1(b)

Suppose you are given that the solution to $y'' + y = 0$ is $y_h = c_1 \cos(x) + c_2 \sin(x)$.

Use variation of parameters to find y_p for

$$y'' + y = \boxed{\sin(x)}$$

$\boxed{b(x)}$

Then state the general solution

Know: $y_1 = \cos(x)$
$y_2 = \sin(x)$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

$$= \cos^2(x) + \sin^2(x)$$

$$= 1$$

Then,

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx$$

$$= \int \frac{-\sin(x) \cdot \sin(x)}{1} dx$$

$$= -\int \sin^2(x) dx$$

$$= -\int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx$$

$$= -\left[\frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right)\right]$$

$$= -\frac{1}{2}x + \frac{1}{4}\sin(2x)$$

And,

$$v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)}\,dx = \int \frac{\cos(x)\sin(x)}{1}\,dx$$

$$= \int \cos(x)\underbrace{\sin(x)}_{u=\sin(x)}\,dx$$

$$du = \cos(x)\,dx$$

$$= \int u\,du = \frac{u^2}{2} = \frac{\sin^2(x)}{2}$$

Thus,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(-\frac{1}{2}x + \frac{1}{4}\sin(2x)\right)\cos(x)$$

$$+ \left(\frac{1}{2}\sin^2(x)\right)\sin(x)$$

General solution to $y'' + y = \sin(x)$ is

$$y = y_h + y_p$$
$$= c_1 \cos(x) + c_2 \sin(x)$$
$$+ \left(-\tfrac{1}{2}x + \tfrac{1}{4}\sin(2x)\right)\cos(x)$$
$$+ \tfrac{1}{2}\sin^2(x)\sin(x)$$

$\boxed{\text{HW 10 } 1(a)}$

Given $y_1 = x^4$ solves

$$x^2 y'' - 7xy' + 16y = 0$$

on $I = (0, \infty)$.

(a) Find another linearly independent solution y_2.

(b) State the general solution.

Divide by x^2 to get

$$y'' - \underbrace{\frac{7}{x}}\, y' + \frac{16}{x^2} y = 0$$

$$\boxed{a_1(x) = -\frac{7}{x}}$$

Then,

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^4 \int \frac{e^{-\int -\frac{7}{x}dx}}{(x^4)^2}\,dx$$

$$= x^4 \int \frac{e^{7\int \frac{1}{x}dx}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{7\ln(x)}}{x^8}\,dx$$

$$\boxed{\begin{aligned}\int \frac{1}{x}\,dx &= \ln|x| \\ &= \ln(x) \\ I = (0,\infty),&\ x>0\end{aligned}}$$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8}\,dx$$

$$\boxed{A\ln(B) = \ln(B^A)}$$

$$\boxed{e^{\ln(A)} = A}$$

$$= x^4 \int \frac{x^7}{x^8}\,dx$$

$$= x^4 \int \frac{1}{x}\,dx = x^4 \ln(x)$$

So, $\boxed{y_2 = x^4 \ln(x)}$

The general solution is

$$y_h = c_1 y_1 + c_2 y_2$$
$$= c_1 x^4 + c_2 x^4 \ln(x)$$

HW 11
1(a)

Find a power series for $f(x) = x^3 + x$ centered at $x_0 = 1$.

$f(x) = x^3 + x \longleftarrow f(1) = 1^3 + 1 = 2$

$f'(x) = 3x^2 + 1 \longleftarrow f'(1) = 3(1)^2 + 1 = 4$

$f''(x) = 6x \longleftarrow f''(1) = 6(1) = 6$

$f'''(x) = 6 \longleftarrow f'''(1) = 6$

$f''''(x) = 0$ — 0 from this point on

$f''''(1) = 0$ — all 0

$\vdots$

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \underbrace{\frac{f''''(1)}{4!}(x-1)^4 + \dots}_{\text{all } 0}$$

$$\underbrace{x^3 + x}_{f(x)} = 2 + 4(x-1) + \frac{6}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3$$

$$\boxed{x^3 + x = 2 + 4(x-1) + 3(x-1)^2 + (x-1)^3}$$

HW 12 #1

Find the first 4 non-zero terms of a power series solution to

$$\boxed{\begin{array}{c} y'' - (x+1)y' + x^2 y = 0 \\ y'(0) = 1, \quad y(0) = 1 \end{array}}$$

What's the radius of convergence?

$$\boxed{x_0 = 0}$$

coefficients

$$-(x+1) = -1 - x + 0x^2 + 0x^3 + \cdots$$
$$x^2 = 0 + 0x + 1 \cdot x^2 + 0x^3 + \cdots$$
$$0 = 0 + 0x + 0x^2 + 0x^3 + \cdots$$

these are all polynomials so they all have $r = \infty$

Our solution will have radius of convergence $r = \infty$

Let's find the solution.

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

Given: $\boxed{y(0)=1}$ $\boxed{y'(0)=1}$

Know $\rightarrow$ $y'' - (x+1)y' + x^2 y = 0$

$$y'' = (x+1)y' - x^2 y$$

$$y''(0) = (0+1)\left[y'(0)\right] - 0^2\left[y(0)\right]$$

$$= 1 \cdot 1 - 0 \cdot 1 = 1$$

$$y''(0) = 1$$

Differentiate to get

$$y''' = (1)y' + (x+1)y'' - 2xy - x^2 y'$$

$$y'''(0) = \left[y'(0)\right] + (0+1)\left[y''(0)\right]$$

$$\cancel{-2(0)\left[y(0)\right]} - \cancel{0^2\left[y'(0)\right]}$$

$$= 1 + 1 \cdot 1 - 0 - 0 = 2$$

$$y'''(0) = 2$$

So,

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots$$

$$y(x) = 1 + 1 \cdot x + \frac{1}{2!}x^2 + \frac{2}{3!}x^3 + \ldots$$

$$\boxed{y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots}$$