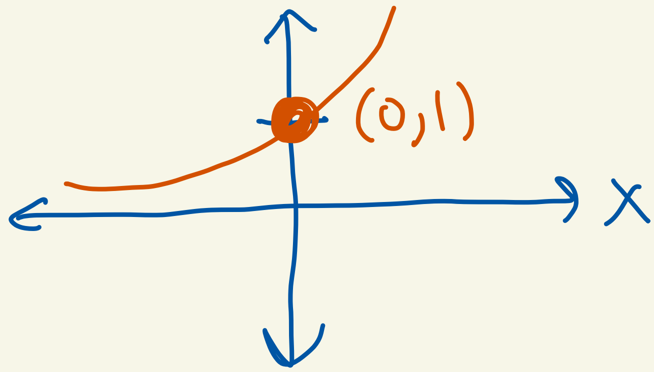
Math 2150-02
1/29/25

Ex: Let's find a solution to the initial-value problem

$$\boxed{\begin{aligned} y' &= y^2 \\ y(0) &= 1 \end{aligned}}$$

non-linear ODE to solve

Condition the solution



Consider $f(x) = \dfrac{1}{1-x}$

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① Let's verify that f satisfies the ODE.

$$f(x) = (1-x)^{-1}$$

$$f'(x) = -(1-x)^{-2} \cdot (-1)$$
$$= (1-x)^{-2} = \frac{1}{(1-x)^2}$$

Note: $f'(x) = \frac{1}{(1-x)^2} = \left[\frac{1}{(1-x)}\right]^2 = [f(x)]^2$

So, f satisfies $y' = y^2$.

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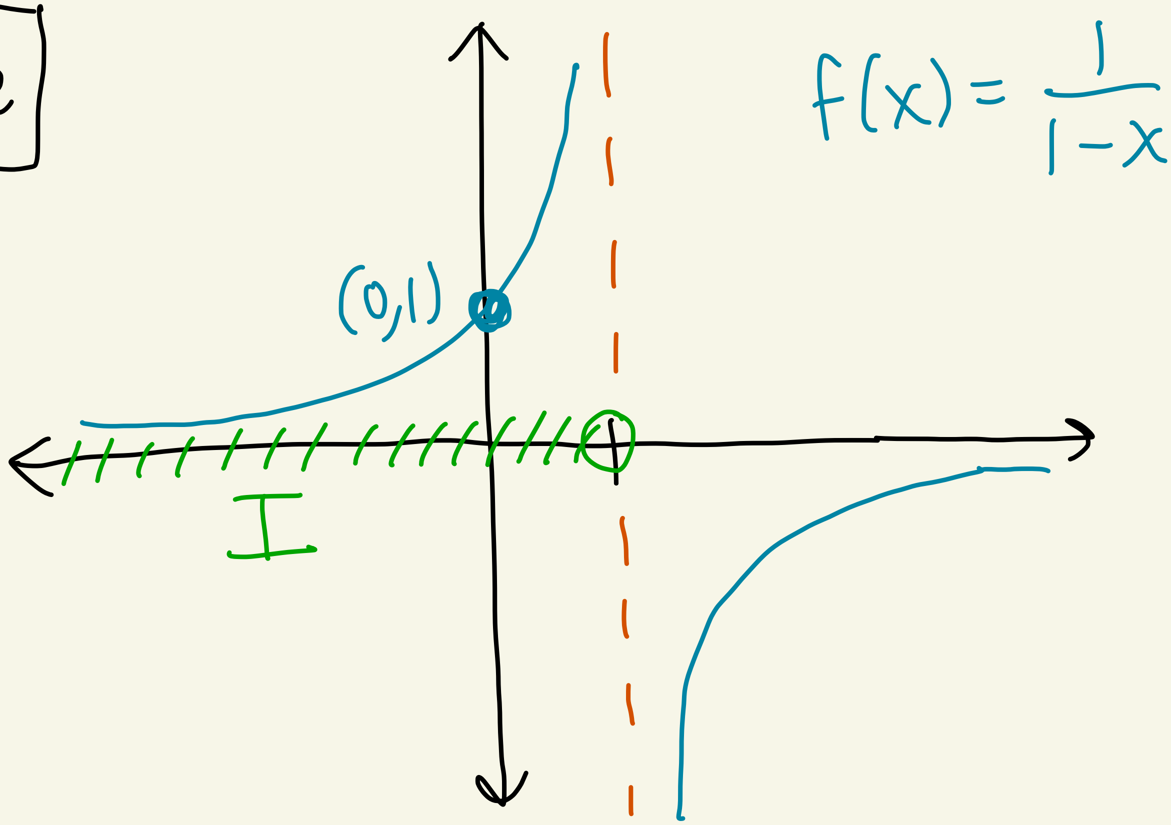
2 $f(0) = \frac{1}{1-0} = 1$

So, f satisfies the initial value problem.

$\boxed{\text{picture}}$



You could say that f satisfies the initial-value problem on $I = (-\infty, 1)$

END OF EXAMPLE

$\boxed{\text{HW 1} \\ 2(d,e)}$

2(d) Let c_1, c_2 be any constants.

Show that
$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

satisfies
$$y'' - 4y = 0$$

on $I = (-\infty, \infty)$.

We have:

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$
$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$
$$f''(x) = 4c_1 e^{2x} + 4c_2 e^{-2x}$$

these are all defined for all x that is on $I = (-\infty, \infty)$

Plug $y=f$, $y''=f''$ into get:

$$y''-4y=(4c_1e^{2x}+4c_2e^{-2x})$$
$$-4(c_1e^{2x}+c_2e^{-2x})$$

$$=0$$

Thus, $f(x)=c_1e^{2x}+c_2e^{-2x}$ satisfies $y''-4y=0$ on $I=(-\infty,\infty)$.

$\boxed{\text{END OF 2(d)}}$

$\boxed{2(e)}$ Find c_1, c_2 so that

$$f(x)=c_1e^{2x}+c_2e^{-2x}$$

satisfies the initial-value problem

$$\boxed{\begin{aligned} & y'' - 4y = 0 \\ & y'(0) = 0 \\ & y(0) = 1 \end{aligned}}$$

ODE

2 conditions on the solution

We saw in 2(d) that

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

satisfies $y'' - 4y = 0$.

Let's find c_1, c_2 so that $f'(0) = 0$ and $f(0) = 1$.

We have

$$f(x) = c_1 e^{2x} + c_2 e^{-2x}$$

$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

We get

$$\boxed{\begin{array}{l} f(0)=1 \\ f'(0)=0 \end{array}} \longrightarrow \boxed{\begin{array}{l} c_1 e^{2(0)} + c_2 e^{-2(0)} = 1 \\ 2c_1 e^{2(0)} - 2c_2 e^{-2(0)} = 0 \end{array}}$$

$e^0 = 1$

$$\boxed{\begin{array}{l} c_1 + c_2 = 1 \\ 2c_1 - 2c_2 = 0 \end{array}} \longrightarrow \boxed{\begin{array}{ll} c_1 + c_2 = 1 & (1) \\ c_1 - c_2 = 0 & (2) \end{array}}$$

(2) gives $\boxed{c_1 = c_2}$.

Plug $c_1 = c_2$ into (1) to get

$c_2 + c_2 = 1$. So, $2c_2 = 1$.

So, $c_2 = 1/2$.

Thus, $c_1 = c_2 = 1/2$.

So, $f(x) = c_1 e^{2x} + c_2 e^{-2x}$
$$= \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$$

solves the initial-value problem.

Topic 3 – First-order linear ODEs

We will give a method to solve

$$y' + a(x)y = b(x)$$

on some interval I where $a(x), b(x)$ are continuous.

Ex:

$b(x)=x$

$$y' + \underbrace{2x}_{a(x)=2x}y = \overbrace{x}$$

$a(x)=2x$

$I=(-\infty,\infty)$

Since $a(x)$ is continuous we can find an antiderivative

$$A(x) = \int a(x)\,dx$$

So, $A'(x) = a(x)$

Ex:

$$A(x) = \int 2x\,dx = x^2$$

Multiply the ODE by $e^{A(x)}$ to get:

$$e^{A(x)}\left(y'+a(x)y\right)=e^{A(x)}b(x)$$

$$\underbrace{e^{A(x)}y'+e^{A(x)}a(x)y}_{\left(e^{A(x)}\cdot y\right)'}=e^{A(x)}b(x)$$

So,

$$\left(e^{A(x)}\cdot y\right)'=e^{A(x)}b(x)$$

Integrate both sides with respect to x to get:

$$e^{A(x)}\cdot y=\int e^{A(x)}b(x)\,dx$$

Thus,

Ex:

$$e^{x^2}y'+2xe^{x^2}y=xe^{x^2}$$

Ex:

$$\left(e^{x^2}\cdot y\right)'=xe^{x^2}$$

Ex:

$$e^{x^2}y=\int xe^{x^2}dx$$

$$y = e^{-A(x)} \cdot \int e^{A(x)} b(x)\,dx$$

Since you can reverse the steps above this is the only solution to the ODE.

Ex:

$$\int x e^{x^2} dx = \frac{1}{2} e^{x^2} + C$$

$$e^{x^2} y = \frac{1}{2} e^{x^2} + C$$

$$y = \frac{1}{2} \underbrace{e^{-x^2} e^{x^2}}_{e^{-x^2+x^2} = e^0 = 1} + C e^{-x^2}$$

$$y = \frac{1}{2} + C e^{-x^2}$$