Math 2150-02
2/17/25

Another method for topic 5

Let's resolve (from last time):

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= e^x + y \quad (1) \\ \frac{\partial f}{\partial y} &= 2 + x + ye^y \quad (2) \end{aligned}}$$

Integrate (1) with respect to x!

$$f(x,y) = e^x + yx + \underbrace{c(y)}_{\text{constant with respect to } x} \quad (3)$$

Integrate (2) with respect to y!

$$f(x,y) = 2y + xy + \underbrace{\int ye^y\,dy}_{ye^y - e^y} + \underbrace{D(x)}_{\text{constant with respect to } y}$$

So,

$$f(x,y) = 2y + xy + ye^y - e^y + D(x) \quad (4)$$

Set 3 equal to 4 to get

$$e^x + \cancel{y}x + C(y) = 2y + x\cancel{y} + ye^y - e^y + D(x)$$

We get

$$\underbrace{e^x} + \underbrace{C(y)} = \underbrace{2y + ye^y - e^y} + \underbrace{D(x)}$$

Set $C(y) = 2y + ye^y - e^y$

$D(x) = e^x$

Plug into 3 or 4 to get f:

Plug $C(y)$ into 3:

$$f(x,y) = e^x + yx + C(y)$$

$$= e^x + yx + 2y + ye^y - e^y$$

I will post both methods in the HW solutions online

Topic 6 – Theory of second order linear ODEs

We worked on first order now its time for second order.

But before we do this we need the def of linear independence.

Def: Let I be an interval. Let f_1 and f_2 be defined on I. We say that f_1 and f_2 are <u>linearly dependent</u> if either

① $f_1(x) = c f_2(x)$ for all x in I

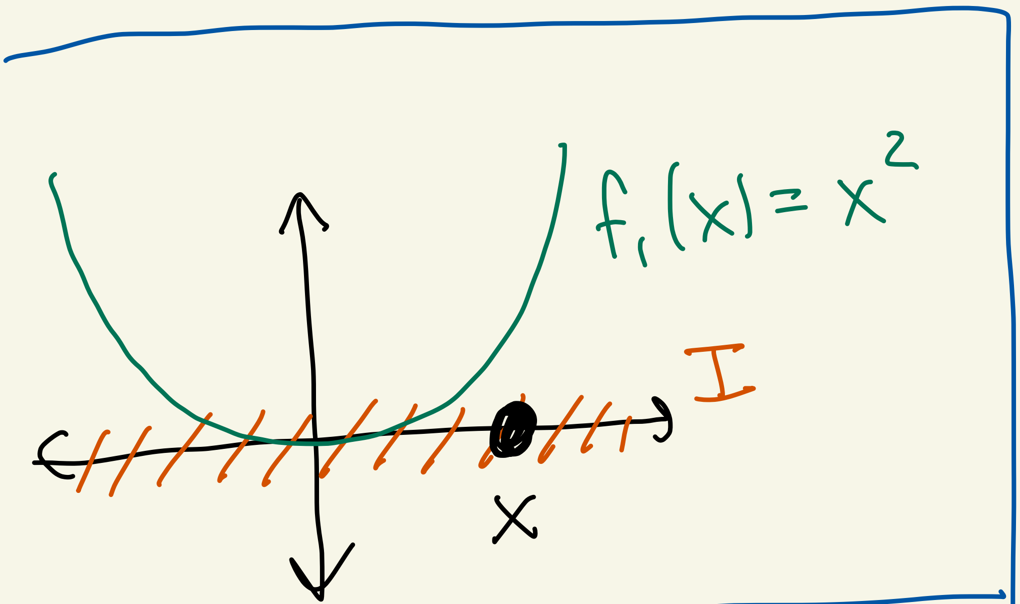
or

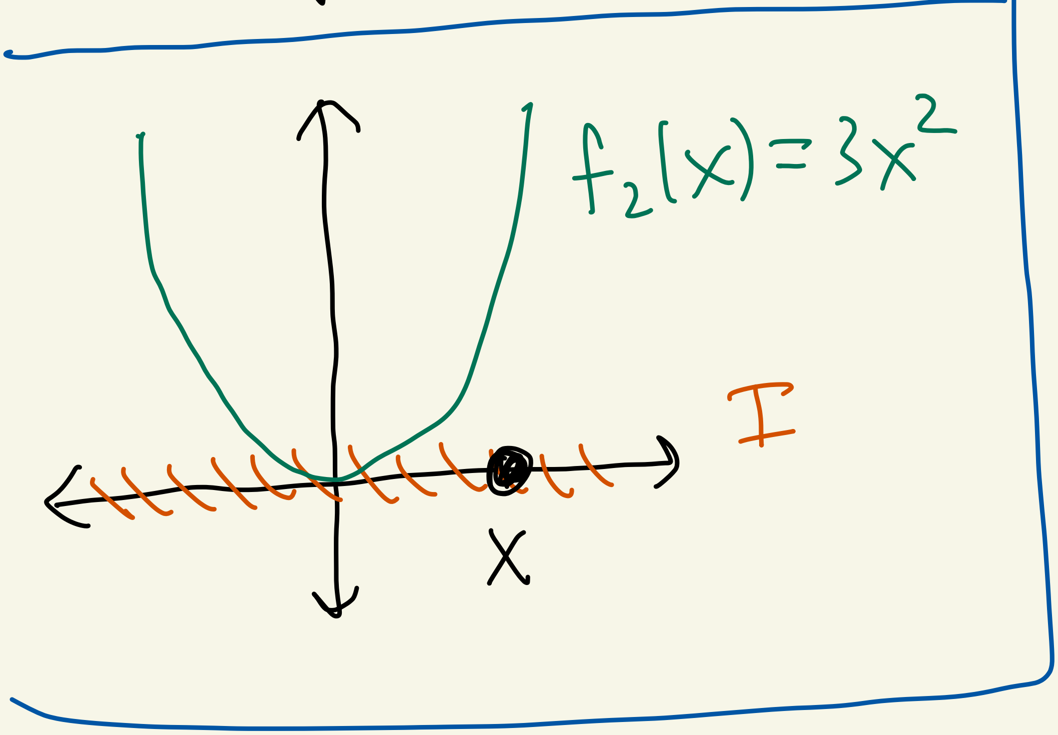
② $f_2(x) = c f_1(x)$ for all x in I

Here c is a constant.

If no such constant exists then f_1, f_2 are called <u>linearly independent</u>.

Ex: Let $f_1(x) = x^2$, $f_2(x) = 3x^2$, $I = (-\infty, \infty)$.





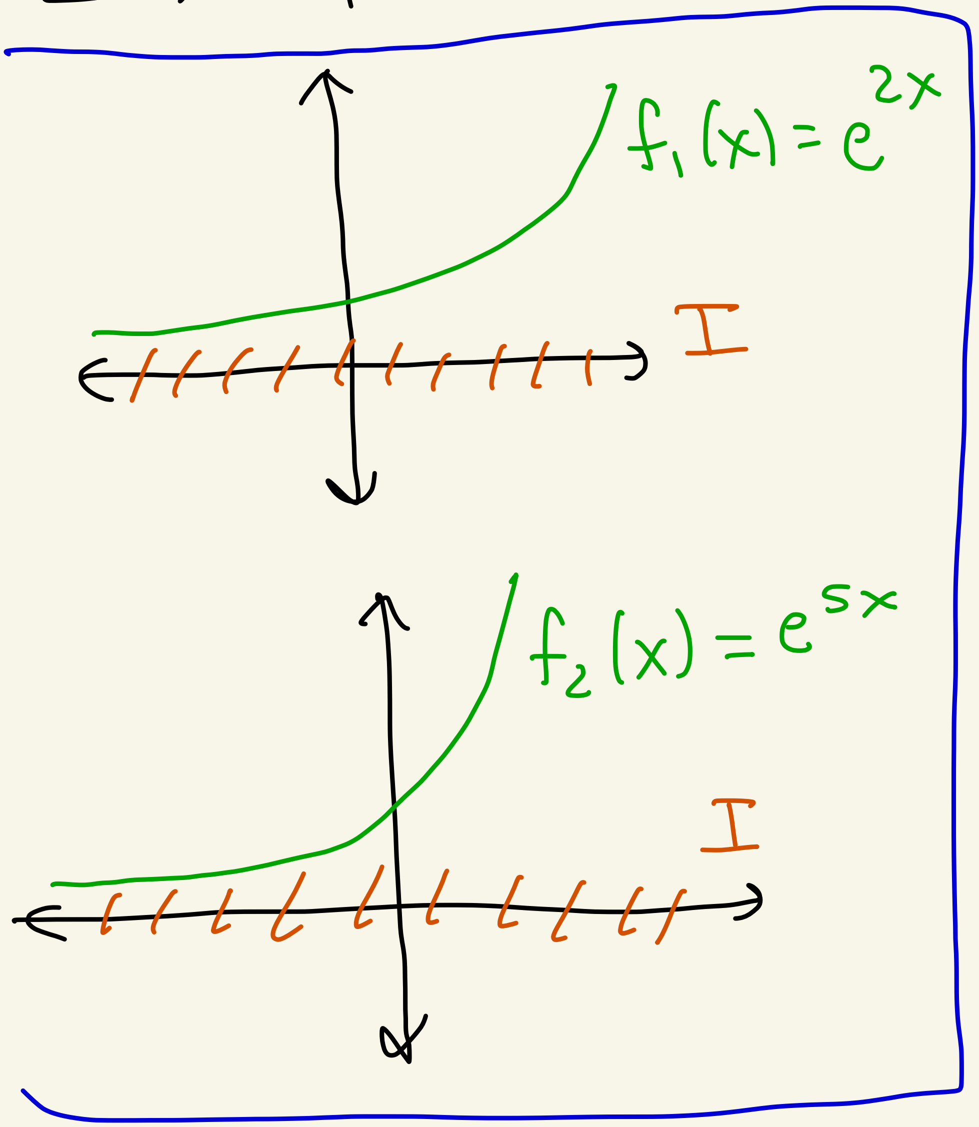
f_1 and f_2 are linearly dependent on I because

$f_2(x) = 3 \cdot f_1(x)$ for all x in I.

or because $f_1(x) = \frac{1}{3} f_2(x)$ for all x in I

Ex: Let $I = (-\infty, \infty)$.

Let $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$.



We will show f_1 and f_2 are linearly independent on I.

Suppose $f_1(x) = c f_2(x)$ for all x in I.

Then, $e^{2x} = c e^{5x}$ for all x.

$x = 0$ would give $1 = c$

$x = 1$ would give $e^2 = c e^5$

That gives $c = e^{-3}$

Then $1 = c = \underbrace{e^{-3}}_{\approx 0.04978\ldots}$ which is nonsense!

Similary there's no way to have $f_2(x) = c f_1(x)$.

f_1 and f_2 are linearly independent

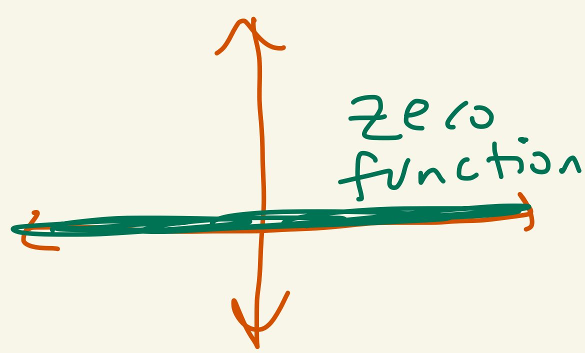
We will learn a method to check for linear independence using the Wronskian.

Josef Wronski (1778-1853)

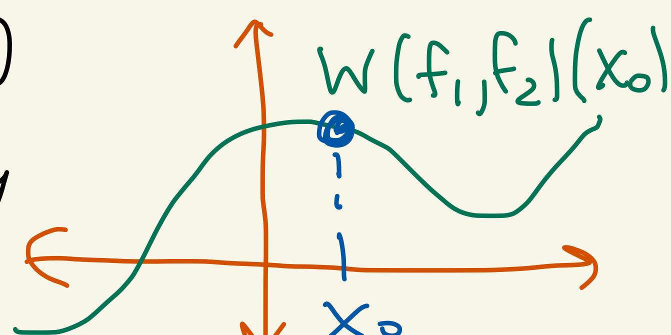
Theorem: Let I be an interval. Let f_1, f_2 be differentiable on I. If the Wronskian

$$W(f_1,f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \underbrace{f_1 f_2' - f_2 f_1'}_{\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}}$$

is not the zero function on I, then f_1, f_2 are linearly independent.



That is, if there exists x_0 in I where $W(f_1,f_2)(x_0) \neq 0$ then f_1, f_2 are linearly independent



<u>Ex:</u> Let $I = (-\infty, \infty)$,
$f_1(x) = e^{2x}$, $f_2(x) = e^{5x}$.

Let's use the Wronskian to show that f_1, f_2 are linearly independent.

We have

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

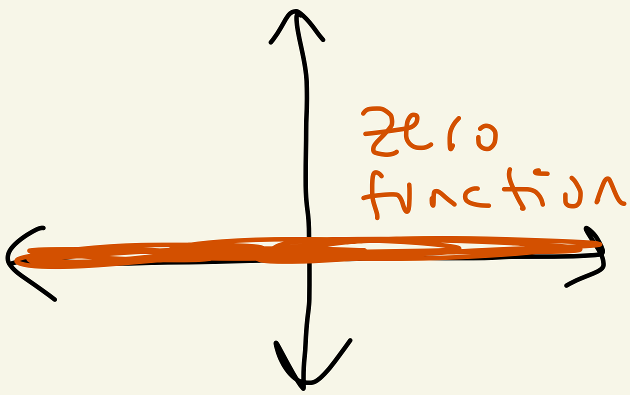
$$= \begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix}$$

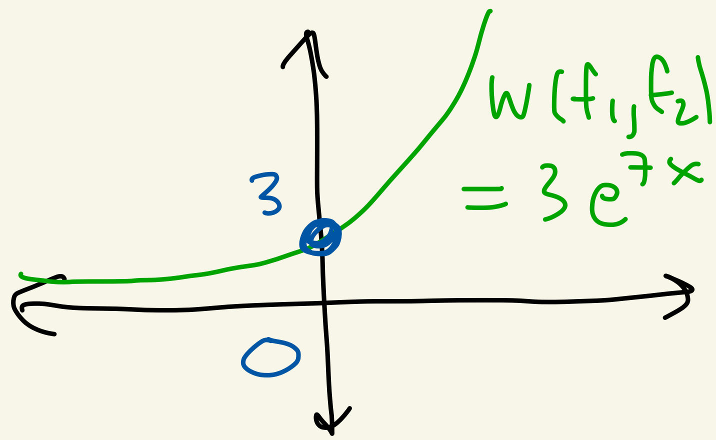
$$= (e^{2x})(5e^{5x}) - (2e^{2x})(e^{5x})$$

$$= 5e^{7x} - 2e^{7x}$$

$$= 3e^{7x}$$

Is this the zero function on I?





No way!!

For example,

$$W(f_1, f_2)(0) = 3e^{7(0)} = 3 \neq 0.$$

Thus, f_1, f_2 are linearly independent on I.

Theorem: Let I be an interval.
Let $a_2(x)$, $a_1(x)$, $a_0(x)$ be continuous on I.
Suppose $a_2(x) \neq 0$ on I.
Consider the homogeneous equation

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = 0} \quad (*)$$

homogeneous when this is 0

If $f_1(x)$ and $f_2(x)$ are linearly independent on I and they both are solutions to $(*)$ on I, then every solution to $(*)$ on I is of the form

$$y_h = c_1 f_1(x) + c_2 f_2(x)$$

where c_1, c_2 are constants.