Math 2150-02
2/5/25

Topic 4 – Separable first order ODEs

<u>Def:</u> A first order ODE is called separable if it is of the form

$$\underbrace{N(y)}_{\text{just \#'s and } y\text{'s}} \cdot y' = \underbrace{M(x)}_{\text{just \#'s and } x\text{'s}}$$

or

$$N(y) \cdot \frac{dy}{dx} = M(x)$$

Ex: $\underbrace{y^3}_{N(y)} \cdot y' = \underbrace{\sin(x)}_{M(x)}$

is separable

Ex: $y' = \frac{x^2}{y}$

Multiply by y to get

$$y \cdot y' = x^2$$

It's separable.

How to solve separable equations

Formal way	Informal way
$N(y) \cdot y' = M(x)$	$N(y) \cdot \frac{dy}{dx} = M(x)$
$\downarrow$	$\downarrow$
$N(y(x)) \cdot y'(x) = M(x)$	$N(y)\,dy = M(x)\,dx$ [differential form notation]
$\downarrow$	$\downarrow$
$\int N(y(x))\, y'(x)\,dx = \int M(x)\,dx$	$\boxed{\int N(y)\,dy = \int M(x)\,dx}$ Now integrate
$\downarrow$ $\boxed{\begin{array}{l} u = y(x) \\ u' = y'(x)dx \end{array}}$	
$\int N(u)\,du = \int M(x)\,dx$ Now integrate	

where $u = y$

Ex: Find a solution to

$$y^2 \frac{dy}{dx} = x - 5$$

Also, what interval I does the solution exist on?

We have

$$y^2 \frac{dy}{dx} = x - 5$$

$$y^2 \, dy = (x-5) \, dx$$

$$\int y^2 \, dy = \int (x-5) \, dx$$

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 - 5x + C$$

$$y^3 = \frac{3}{2}x^2 - 15x + 3C$$

$D = 3C$

$$y^3 = \frac{3}{2}x^2 - 15x + D$$

$$\boxed{y = \left(\frac{3}{2}x^2 - 15x + D\right)^{1/3}}$$

Here $I = (-\infty, \infty)$

Any x is is ok to plug into the formula

Ex: Find a solution to

$$\frac{dy}{dx} + 2xy = 0$$

On what interval I does the solution exist?

You could use topic 3 here but let's not.

We have

$$\frac{dy}{dx} + 2xy = 0$$

$$\frac{dy}{dx} = -2xy$$

$$\frac{1}{y}\,dy = -2x\,dx$$

$$\int \frac{1}{y}\,dy = \int (-2x)\,dx$$

$$\ln|y| = -2\frac{x^2}{2} + C$$

$$\ln|y| = -x^2 + C$$

$$e^{\ln|y|} = e^{-x^2+C}$$

we want the y!

$$|y| = e^{-x^2+C}$$

$$y = \pm e^{-x^2+C}$$

$$y = \pm e^{-x^2} e^{c}$$

$$y = \boxed{\pm e^{c}} e^{-x^2}$$

Constant

$y = De^{-x^2}$ where D is a constant

and $I = (-\infty, \infty)$

the function is defined for any x.

HW #4
1 (c)

Find a solution to

$$\frac{dy}{dx} = \frac{-x}{y}$$

Solve for y if you can.
If we can, find the interval I
the solutions exists on.

We have

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = \int (-x)\,dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$x2\ \Gamma$

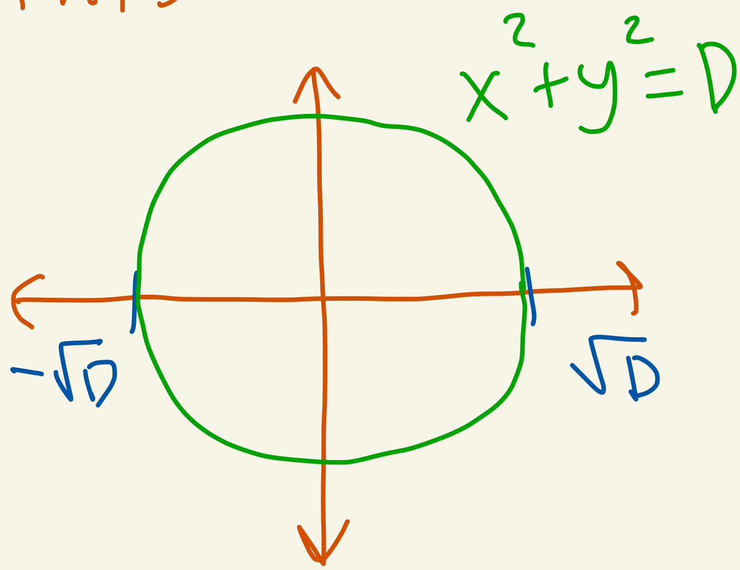
$$y^2 = -x^2 + 2C$$

$$\boxed{D = 2C}$$

$$y^2 = -x^2 + D$$

We could stop here and we'd get an implicit equation for x and y. It's this:

$$x^2 + y^2 = D$$



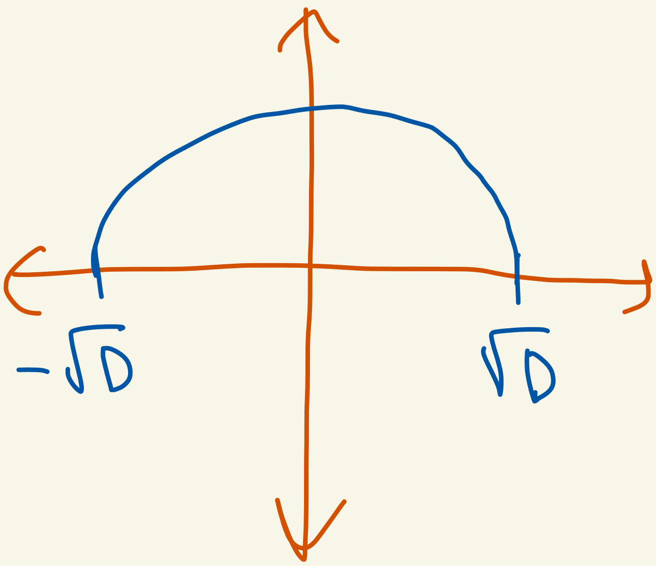
But we <u>can</u> solve for y in $y^2 = -x^2 + D$. We get:

$$y = \pm\sqrt{-x^2 + D}$$

We get two functions:

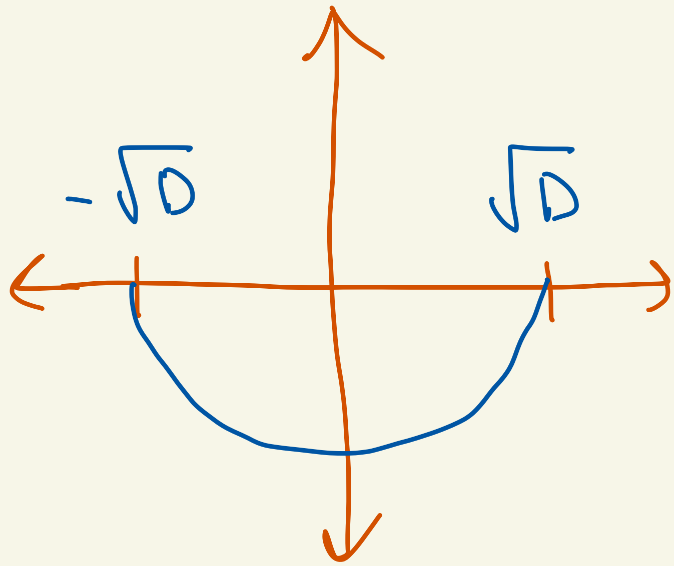
SOLUTION 1

$$y = \sqrt{-x^2 + D}$$



SOLUTION 2

$$y = -\sqrt{-x^2 + D}$$



$$I = [-\sqrt{D}, \sqrt{D}]$$

or

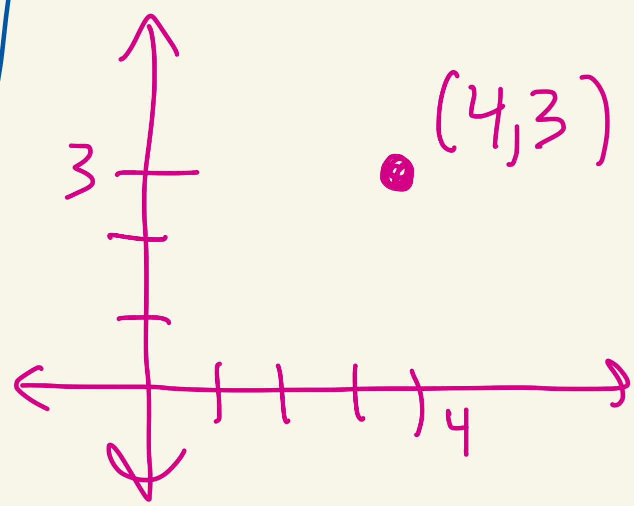
$$I = (-\sqrt{D}, \sqrt{D})$$

for this class we use open intervals

Hw 4 #1(d)

Find a solution to

$$\boxed{\begin{gathered}\frac{dy}{dx} = \frac{-x}{y} \\ y(4) = 3\end{gathered}}$$



the solution must go through (4,3)

We know an implicit solution to

$\frac{dy}{dx} = \frac{-x}{y}$ is $y^2 = -x^2 + D$.

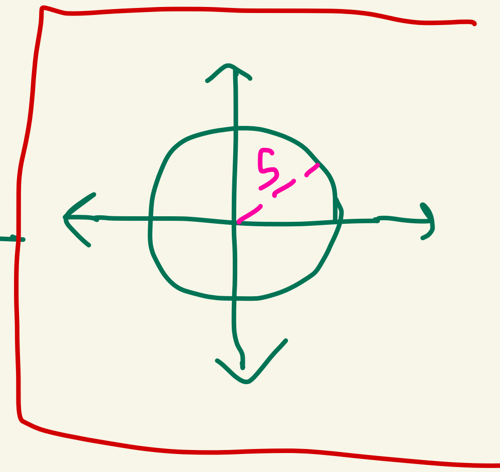
Plug in $x = 4$ and $y = 3$. $\boxed{y(4) = 3}$

We get $(3)^2 = -(4)^2 + D$

So, $9 = -16 + D$

So, $D = 25$

Thus, $y^2 = -x^2 + 25$



So, $y = \pm\sqrt{-x^2+25}$

Want top half so pick + to get:

$$y = \sqrt{-x^2+25}$$

$I = [-5, 5]$