2150-02
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Topic 9 – Variation of parameters

Topic 9 is another method to find y_p.
It works in situations that topic 8 doesn't, such as

$$y'' + y = \tan(x)$$

It will even work when the coefficients are not constants!

Let's derive the formula.

Suppose we want to find a particular solution y_p to

$$\boxed{y'' + a_1(x)\, y' + a_0(x)\, y = b(x)}$$

where $a_1(x)$, $a_0(x)$, $b(x)$ are continuous on an interval I.

We need to have already found the general solution to

$$\boxed{y'' + a_1(x)\, y' + a_0(x)\, y = 0}$$

Let's say its

$$y_h = c_1 y_1(x) + c_2 y_2(x)$$

where y_1, y_2 are linearly independent.

Given y_1, y_2 above set

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1, v_2 will be determined.

<u>Ex:</u> $y'' - y = 0 \longrightarrow y_h = c_1 \underbrace{e^{x}}_{y_1} + c_2 \underbrace{e^{-x}}_{y_2}$

$y'' - y = \sin(x) \longrightarrow y_p = v_1 e^{x} + v_2 e^{-x}$

We want to plug our guess

$$y_p = v_1 y_1 + v_2 y_2$$

into

$$y'' + a_1(x) y' + a_0(x) y = b(x)$$

Let's get the derivatives.

We have

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = v_1' y_1 + v_1 y' + v_2' y_2 + v_2 y_2'$$

$$= (v_1 y_1' + v_2 y_2') + \underbrace{(v_1' y_1 + v_2' y_2)}$$

We will assume $\boxed{v_1' y_1 + v_2' y_2 = 0}$ to make this easier. It turns out it will still work.

So we get

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = v_1 y_1' + v_2 y_2' \quad \leftarrow \boxed{\text{from above assumption}}$$

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Plug this all into

$$y'' + a_1(x) y' + a_0(x) y = b(x)$$

to get:

$$(v_1'y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'') \leftarrow y_p''$$

$$+ a_1(x)(v_1 y_1' + v_2 y_2') \leftarrow + a_1(x) y_p'$$

$$+ a_0(x)(v_1 y_1 + v_2 y_2) \leftarrow + a_0(x) y_p$$

$$= b(x) \leftarrow = b(x)$$

Rearranging:

$$v_1(\underbrace{y_1'' + a_1(x) y_1' + a_0(x) y_1}_{0})$$

$$+ v_2(\underbrace{y_2'' + a_1(x) y_2' + a_0(x) y_2}_{0})$$

$$+ (v_1' y_1' + v_2' y_2') = b(x)$$

The above are 0 because y_1, y_2 solved $y'' + a_1(x) y' + a_0(x) y = 0$

We get

$$\boxed{v_1' y_1' + v_2' y_2' = b(x)}$$

This plus our assumption give us:

$$\boxed{\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \quad (1) \leftarrow \text{assumption above} \\ v_1' y_1' + v_2' y_2' &= b(x) \quad (2) \leftarrow \text{derivation} \end{aligned}}$$

To solve the above calculate $y_1' * (1) - y_1 * (2)$ to get:

$$\cancel{y_1' v_1' y_1} + y_1' v_2' y_2 - \cancel{y_1 v_1' y_1'} - y_1 v_2' y_2' = y_1' \cdot 0 - y_1 \cdot b(x)$$

This gives

$$v_2'(y_1' y_2 - y_1 y_2') = -y_1 b(x)$$

So,

$$v_2' = \frac{-y_1 b(x)}{y_1' y_2 - y_1 y_2'}$$

Thus,

$$v_2 = \int \frac{-y_1 b(x)}{y_1' y_2 - y_1 y_2'} dx$$

$$= \int \frac{y_1 b(x)}{y_1 y_2' - y_1' y_2} dx$$

$$= \boxed{\int \frac{y_1 b(x)}{W(y_1, y_2)} dx}$$

Similarly if you calculate

$$y_2' * 1 - y_2 * 2$$

you can derive

$$\boxed{v_1 = \int \frac{-y_2\, b(x)}{W(y_1, y_2)}\, dx}$$

Summary:

Suppose you have two linearly independent solutions y_1, y_2 to

$$y'' + a_1(x) y' + a_0(x) y = 0$$

Then a particular solution to

$$y'' + a_1(x) y' + a_0(x) y = b(x)$$

is given by

$$y_p = v_1 y_1 + v_2 y_2$$

where

$$v_1 = \int \frac{-y_2\, b(x)}{W(y_1, y_2)}\, dx \, , \quad v_2 = \int \frac{y_1\, b(x)}{W(y_1, y_2)}\, dx$$
