Math 2150-02
3/24/25

Recap from last time

Given: y_1, y_2 are linearly independent solutions to

$$y'' + a_1(x)y' + a_0(x)y = 0$$

Result: A particular solution to

$$y'' + a_1(x)y' + a_0(x)y = b(x)$$

is given by

$$y_p = v_1 y_1 + v_2 y_2$$

where

$$v_1 = \int -\frac{y_2\, b(x)}{W(y_1, y_2)}\,dx, \quad v_2 = \int \frac{y_1\, b(x)}{W(y_1, y_2)}\,dx$$

Ex: Solve

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

using variation of parameters

Step 1: Solve

$$y'' - 4y' + 4y = 0$$

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) = 0$$

$$r = 2 \leftarrow \text{repeated root}$$

$$y_h = c_1 e^{2x} + c_2 x e^{2x}$$

This gives us

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}$$

to use in the next step

Step 2: Now we find y_p for

$$y'' - 4y' + 4y = \underbrace{(x+1)e^{2x}}_{b(x)}$$

We use

$$y_p = v_1 y_1 + v_2 y_2$$

where

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}$$

Let's calculate v_1 and v_2.

We need the Wronskian!

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + x(2e^{2x}) \end{vmatrix}$$

$$= (e^{2x})(e^{2x} + 2xe^{2x})$$

$$- (xe^{2x})(2e^{2x})$$

$$= e^{4x} + \cancel{2xe^{4x}} - \cancel{2xe^{4x}}$$

$$= e^{4x}$$

Then,

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx$$

$$= \int \frac{-\left(xe^{2x}\right)\cdot(x+1)e^{2x}}{e^{4x}}\,dx$$

$$= \int \frac{-x(x+1)e^{2x}e^{2x}}{e^{4x}}\,dx$$

$$= \int \frac{(-x^2-x)\cancel{e^{4x}}}{\cancel{e^{4x}}}\,dx$$

$$= \int (-x^2-x)\,dx$$

$$= \boxed{-\frac{x^3}{3} - \frac{x^2}{2}}$$

And,

$$v_2 = \int \frac{y_1\, b(x)}{W(y_1, y_2)}\, dx$$

$$e^{2x} e^{2x} = e^{4x}$$

$$= \int \frac{\cancel{e^{2x}} \cdot (x+1) \cancel{e^{2x}}}{\cancel{e^{4x}}}\, dx$$

$$= \int (x+1)\, dx$$

$$= \boxed{\frac{1}{2}x^2 + x}$$

Thus,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right) e^{2x} + \left(\frac{1}{2}x^2 + x\right) x e^{2x}$$

Step 3: The general solution to

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

is

$$y = y_h + y_p$$

$$\boxed{y = c_1 e^{2x} + c_2 x e^{2x} + \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}}$$

<u>Ex:</u> Let's solve

$$y'' + y = \tan(x)$$

<u>Step 1:</u> Solve

$$y'' + y = 0$$

The characteristic equation is

$$r^2 + 1 = 0$$

roots are

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{\pm\sqrt{-4}}{2} = \frac{\pm\sqrt{\cancel{4}}\sqrt{-1}}{\cancel{2}}$$

$$= \pm\sqrt{-1} = \pm i$$

$$= \underbrace{0 \pm 1 \cdot i}_{\alpha \pm \beta i}$$

So,

$$y_h = \underbrace{c_1 e^{0x} \cos(1 \cdot x) + c_2 e^{0x} \sin(1 \cdot x)}_{c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)}$$

$$\boxed{y_h = c_1 \cos(x) + c_2 \sin(x)}$$

$$\boxed{e^{0x} = e^{0} = 1}$$

For next step

$y_1 = \cos(x)$, $y_2 = \sin(x)$

Step 2: Find y_p for

$$y'' + y = \underbrace{\tan(x)}_{b(x)}$$

Use $y_1 = \cos(x)$, $y_2 = \sin(x)$.

We will need

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= \cos(x)\cos(x) - (\sin(x))(-\sin(x))$$

$$= \cos^2(x) + \sin^2(x)$$

$$= 1$$

We get

$$V_1 = \int \frac{-y_2\, b(x)}{W(y_1, y_2)}\, dx$$

$$= \int \frac{-\sin(x)\ \tan(x)}{1}\, dx$$

$$= \int -\sin(x) \cdot \frac{\sin(x)}{\cos(x)}\, dx$$

$$= \int \frac{-\sin^2(x)}{\cos(x)}\, dx$$

$$= \int \frac{\cos^2(x) - 1}{\cos(x)}\, dx$$

$$\boxed{\sin^2(x) + \cos^2(x) = 1}$$

$$= \int \left(\frac{\cos^2(x)}{\cos(x)} - \frac{1}{\cos(x)} \right) dx$$

$$= \int \left(\cos(x) - \sec(x) \right) dx$$

$$= \underbrace{\boxed{\sin(x) - \ln\left| \sec(x) + \tan(x) \right|}}_{V_1}$$

And,

$$V_2 = \int \frac{y_1\, b(x)}{W(y_1, y_2)} dx$$

$$= \int \frac{\cos(x) \tan(x)}{1} dx$$

$$= \int \cancel{\cos(x)} \cdot \frac{\sin(x)}{\cancel{\cos(x)}} dx$$

$$= \int \sin(x)\,dx = \underbrace{\boxed{-\cos(x)}}_{v_2}$$

Thus,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(\sin(x) - \ln\left|\sec(x) + \tan(x)\right|\right)\cos(x)$$

$$- \cos(x) \cdot \sin(x)$$

$$\boxed{y_p = -\ln\left|\sec(x) + \tan(x)\right| \cdot \cos(x)}$$

Step 3: The general solution to $y'' + y = \tan(x)$ is

$$y = y_h + y_p$$

$$= c_1 \cos(x) + c_2 \sin(x)$$

$$- \ln|\sec(x) + \tan(x)) \cdot \cos(x)$$