Math 2150-02
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Topic 8 – The method of undetermined coefficients

We want to be able to solve equations of the form:

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

where a_2, a_1, a_0 are constants.

<u>Method:</u> (From topics 6 and 7)

① Find the general solution y_h to

$$a_2 y'' + a_1 y' + a_0 y = 0$$

② Find a particular solution y_p to

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

③ Then the general solution to

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

is $y = y_h + y_p$

① was topic 7.
② is now, topic 8.
③ is from topic 6.

For topic 8 we guess y_p based on what $b(x)$ is. There's a table in the "cheat sheet" handout to do this. It's this:

Undetermined coefficients guess for y_p:

$b(x)$	y_p
constant	A
$5x - 3$	$Ax + B$
$10x^2 - x + 1$	$Ax^2 + Bx + C$
$\sin(6x)$	$A\cos(6x) + B\sin(6x)$
$\cos(6x)$	$A\cos(6x) + B\sin(6x)$
e^{3x}	Ae^{3x}
$(2x+1)e^{3x}$	$(Ax + B)e^{3x}$
x^2e^{3x}	$(Ax^2 + Bx + C)e^{3x}$
$e^{3x}\sin(4x)$	$Ae^{3x}\cos(4x) + Be^{3x}\sin(4x)$
$e^{3x}\cos(4x)$	$Ae^{3x}\cos(4x) + Be^{3x}\sin(4x)$
$5x^2\sin(4x)$	$(Ax^2 + Bx + C)\cos(4x) + (Dx^2 + Ex + F)\sin(4x)$

Ex: Find the general solution to:

$$y'' + 3y' + 2y = 2x^2$$

Step 1: Solve

$$y'' + 3y' + 2y = 0$$

The characteristic equation is

$$r^2 + 3r + 2 = 0$$

Factoring way

$$(r+1)(r+2) = 0$$

$$r = -1, -2$$

The roots are:

$$r = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{1}}{2} = \frac{-3 \pm 1}{2}$$

$$= \frac{-3+1}{2}, \frac{-3-1}{2} = -1, -2$$

So, $y_h = c_1 e^{-x} + c_2 e^{-2x}$

Step 2: Guess a solution y_p to

$$y'' + 3y' + 2y = \underbrace{2x^2}_{b(x)}$$

We guess $y_p = Ax^2 + Bx + C$.

We have $y_p' = 2Ax + B$

and $y_p'' = 2A$

Plug these into the equation to get

$$\underbrace{(2A)}_{y_p''} + 3\underbrace{(2Ax+B)}_{y_p'} + 2\underbrace{(Ax^2+Bx+C)}_{y_p} = 2x^2$$

Simplify the left side

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = 2x^2$$

Group like terms:

$$\underbrace{\boxed{2A}}_{2}x^2 + \underbrace{\boxed{(6A+2B)}}_{0}x + \underbrace{\boxed{(2A+3B+2C)}}_{0} = 2x^2$$

Need
$$\boxed{\begin{aligned} 2A &= 2 \quad (1)\\ 6A+2B &= 0 \quad (2)\\ 2A+3B+2C &= 0 \quad (3)\end{aligned}}$$

(1) gives $\boxed{A=1.}$

Plug $A=1$ into (2) to get:

$$6(1)+2B=0$$
$$\boxed{B=-3}$$

Plug $A=1$, $B=-3$ into (3) to get:

$$2(1)+3(-3)+2C=0$$
$$\boxed{C=7/2}$$

So,
$$\boxed{y_p = Ax^2+Bx+C = x^2-3x+7/2}$$

Step 3: Thus, the general solution to

$$y''+3y'+2y=2x^2$$

is

$$y = y_h + y_p$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + x^2 - 3x + 7/2}$$

Ex: Solve

$$y'' - y' + y = 2\sin(3x)$$

Step 1: Solve

$$y'' - y' + y = 0$$

The characteristic equation is

$$r^2 - r + 1 = 0$$

The roots are

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm \sqrt{3}\sqrt{-1}}{2} = \frac{1 \pm \sqrt{3}\, i}{2}$$

$$= \underbrace{\frac{1}{2} \pm \frac{\sqrt{3}}{2} i}_{\alpha \pm \beta i}$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{\sqrt{3}}{2}$$

Formula:

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

So,

$$\boxed{y_h = c_1 e^{x/2} \cos\left(\frac{\sqrt{3}}{2} x\right) + c_2 e^{x/2} \sin\left(\frac{\sqrt{3}}{2} x\right)}$$

Step 2: Now find y_p for

$$y'' - y' + y = \underbrace{2\sin(3x)}_{b(x)}$$

Guess this:

$$y_p = A\sin(3x) + B\cos(3x)$$

We have

$$y_p' = 3A\cos(3x) - 3B\sin(3x)$$

$$y_p'' = -9A\sin(3x) - 9B\cos(3x)$$

Plug these into the equation:

$$\left.\begin{aligned}&(-9A\sin(3x) - 9B\cos(3x))\\ &-(3A\cos(3x) - 3B\sin(3x))\\ &+(A\sin(3x) + B\cos(3x))\end{aligned}\right\} \begin{aligned}&y_p''\\ &-y_p'\\ &+y_p\end{aligned}$$

$$= 2\sin(3x)$$

Gather like terms:

$$\underbrace{(-8A+3B)}_{2}\sin(3x)+\underbrace{(-3A-8B)}_{0}\cos(3x) = 2\sin(3x)$$

Need: $\boxed{\begin{array}{l} -8A+3B=2 \\ -3A-8B=0 \end{array}}$ ① ②

In ② we get $A = \frac{-8}{3}B$.

Plug this into ① to get:

$$-8\left(-\frac{8}{3}B\right)+3B=2$$

So, $\frac{73}{3}B = 2$

So, $B = \frac{6}{73}$

Then $A = \frac{-8}{3}B = \frac{-8}{3}\left(\frac{6}{73}\right) = \frac{-16}{73}$

So,

$$y_p = \frac{-16}{73}\sin(3x) + \frac{6}{73}\cos(3x)$$

Step 3: The general solution to

$$y'' - y' + y = 2\sin(3x)$$

is

$$y = y_h + y_p$$

$$= c_1 e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$-\frac{16}{73}\sin(3x) + \frac{6}{73}\cos(3x)$$