2150-02
3/5/25

(topic 8 continued...)

What can go wrong with the guessing method for y_p that we learned?

If your y_p guess contains terms that appear in y_h then you will have to multiply those terms by the smallest power x^n that removes the duplication with the y_h term

Ex: Solve

$$y'' - 5y' + 4y = 8e^x$$

Step 1: Solve the homogeneous equation

$$y'' - 5y' + 4y = 0$$

The characteristic equation is

$$r^2 - 5r + 4 = 0$$

The roots are

$$r = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2}$$

$$= \frac{5+3}{2}, \frac{5-3}{2} = \boxed{4, 1}$$

The general solution to

$$y'' - 5y' + 4y = 0$$

is

$$\boxed{y_h = c_1 e^{4x} + c_2 e^{x}}$$

<u>Step 2:</u> Now we guess y_p for

$$y'' - 5y' + 4y = 8e^{x}$$

Our table says to guess

$$y_p = Ae^{x}$$

This won't work, let's try plugging it in.

$$y_p = Ae^{x}, \quad y_p' = Ae^{x}, \quad y_p'' = Ae^{x}$$

Plug into $y'' - 5y' + 4y$ to get

We get

$$y_p'' - 5y_p' + 4y_p = Ae^x - 5(Ae^x) + 4(Ae^x)$$
$$= 0$$

this isn't $8e^x$

Why didn't our guess $y_p = Ae^x$ work? Because y_p is a term in $y_h = c_1 e^{4x} + \underbrace{c_1 e^x}_{\text{this is } Ae^x}$

How do we fix this?
Multiply by x.
Guess instead: $\boxed{y_p = Axe^x}$

Let's try it.

$$y_p = Axe^x$$
$$y_p' = Ae^x + Axe^x$$
$$y_p'' = Ae^x + Ae^x + Axe^x$$
$$= 2Ae^x + Axe^x$$

Now plug into $y'' - 5y' + 4y = 8e^x$ and get:

$$\underbrace{(2Ae^x + Axe^x)}_{y_p''} - 5\underbrace{(Ae^x + Axe^x)}_{y_p'} + 4\underbrace{Axe^x}_{y_p} = 8e^x$$

We get,

$$2Ae^x + Axe^x - 5Ae^x - 5Axe^x + 4Axe^x = 8e^x$$

So,

$$-3Ae^x = 8e^x$$

Need $-3A = 8$.

So, $A = -8/3$.

$$\boxed{\text{Thus, } y_p = \frac{-8}{3}xe^x \text{ is a particular solution to } y'' - 5y' + 4y = 8e^x}$$

Step 3: The general solution to

$$y'' - 5y' + 4y = 8e^{x}$$

is

$$y = y_h + y_p$$

$$= \boxed{c_1 e^{4x} + c_2 e^{x} - \frac{8}{3} x e^{x}}$$

Ex: Solve

$$y'' - 2y' + y = e^x$$

Step 1: Solve

$$y'' - 2y' + y = 0$$

The characteristic polynomial is

$$r^2 - 2r + 1 = 0$$

$$r^2 - 2r + 1 = 0$$
$$(r-1)(r-1) = 0$$
$$r = 1$$

The roots are:

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{0}}{2} = \boxed{1}$$

The general solution is

$$\boxed{y_h = c_1 e^x + c_2 x e^x}$$

Step 2: Now we must guess a solution to

$$y'' - 2y' + y = e^x$$

The table says to guess

$$y_p = Ae^x$$

But this is a term in y_h.
Bump up by x. Try:

$$y_p = Axe^x$$

But this also appears in y_h.
Bump up by x again.
Try

$$y_p = Ax^2e^x$$

This does not appear in y_h.
We are good to plug it in.

We have

$$y_p = Ax^2e^x$$

$$y_p' = 2Axe^x + Ax^2e^x$$

$$y_p'' = (2Ae^x + 2Axe^x) + (2Axe^x + Ax^2e^x)$$

$$= 2Ae^x + 4Axe^x + Ax^2e^x$$

Plug this into $\boxed{y'' - 2y' + y = e^x}$ to get:

$$\underbrace{(2Ae^x + 4Axe^x + Ax^2e^x)}_{y_p''} - 2\underbrace{(2Axe^x + Ax^2e^x)}_{y_p'}$$

$$+ \underbrace{Ax^2e^x}_{y_p} = e^x$$

This becomes

$$2Ae^x + 4Axe^x + Ax^2e^x - 4Axe^x - 2Ax^2e^x$$

$$+ Ax^2e^x = e^x$$

We get:

$$2Ae^x = e^x$$

Need

$$2A = 1$$

or

$$A = 1/2$$

Thus,

$$y_p = \frac{1}{2}x^2e^x$$

is our solution to $y'' - 2y' + y = e^x$

Step 3: The general solution to $y'' - 2y' + y = e^x$ is

$$y = y_h + y_p$$

$$= \boxed{c_1e^x + c_2xe^x + \frac{1}{2}x^2e^x}$$

Ex: How would you guess y_p for

$$y'' + 2y' = \underbrace{2x + 5}_{\text{table says guess } Ax+B} - \underbrace{e^x}_{\text{table says guess } Ce^x} \quad ?$$

What you would guess here is

$$y_p = Ax + B + Ce^x$$

add the two guesses