Math 2150-02
4/14/25

Schedule

4/14 TOPIC 12	4/16 TOPIC 12
4/21 REVIEW	4/23 REVIEW
4/28 TEST 2	4/30 TOPIC 13
5/5 TOPIC 13 (HAND BACK TESTS)	5/7 REVIEW FOR FINAL
5/12	5/14 FINAL 12-2

$\boxed{\text{HW 11}}$

$\boxed{1(g)}$ Find the first few terms for the power series of $f(x) = \frac{1}{x^2}$ centered at $x_0 = 1$

Use Taylor series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \ldots$$

Here $x_0 = 1$.

$$f(x) = \frac{1}{x^2} = x^{-2} \longleftarrow f(1) = \frac{1}{1^2} = 1$$

$$f'(x) = -2x^{-3} \longleftarrow f'(1) = \frac{-2}{(1)^3} = -2$$

$$= {}^{-2}/_{x^3}$$

$$f''(x) = 6x^{-4} = \frac{6}{x^4} \longleftarrow f''(1) = \frac{6}{(1)^4} = 6$$

$$f'''(x) = -24x^{-5} = -\frac{24}{x^5} \longleftarrow f'''(1) = \frac{-24}{(1)^5} = -24$$

$\boxed{\text{Plug into Taylor's series formula}}$:

$$f(x) = 1 - 2(x-1) + \frac{6}{2!}(x-1)^2 - \frac{24}{3!}(x-1)^3 + \dots$$

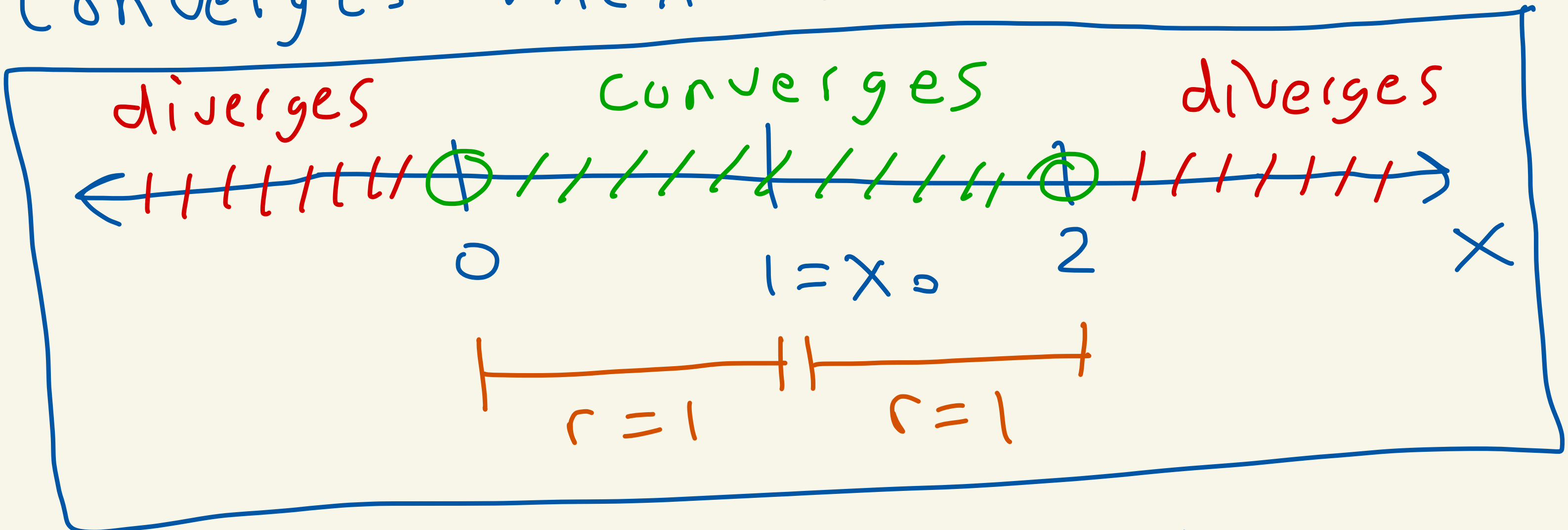
$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \dots$$

This method though doesn't provide the radius of convergence.

Last time we saw that:

$$\frac{1}{x} = 1-(x-1)+(x-1)^2-(x-1)^3+\dots$$

converges when $0<x<2$



radius of convergence $r=1$.

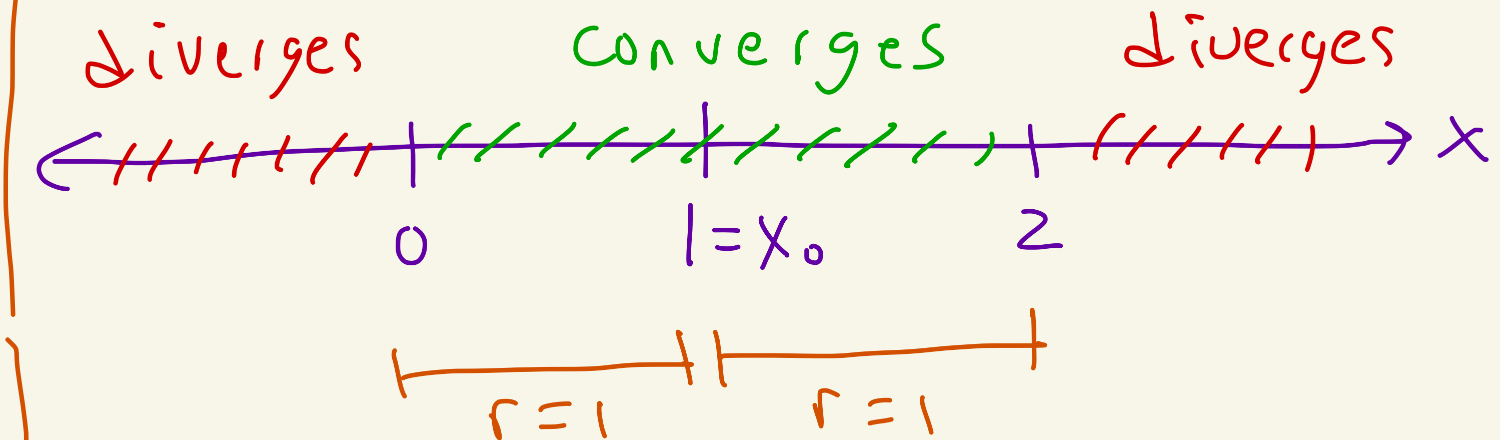
Differentiate both sides above and the radius of convergence stays the same at $r=1$.

$$\underbrace{-\frac{1}{x^2}}_{\text{derivative of } \frac{1}{x}=x^{-1}} = 0-1+2(x-1)-3(x-1)^2+\dots$$

$$-\frac{1}{x^2} = -1 + 2(x-1) - 3(x-1)^2 + \cdots$$

$$\boxed{\frac{1}{x^2} = 1 - 2(x-1) + 3(x-1)^2 - \cdots}$$

radius of convergence $r = 1$.

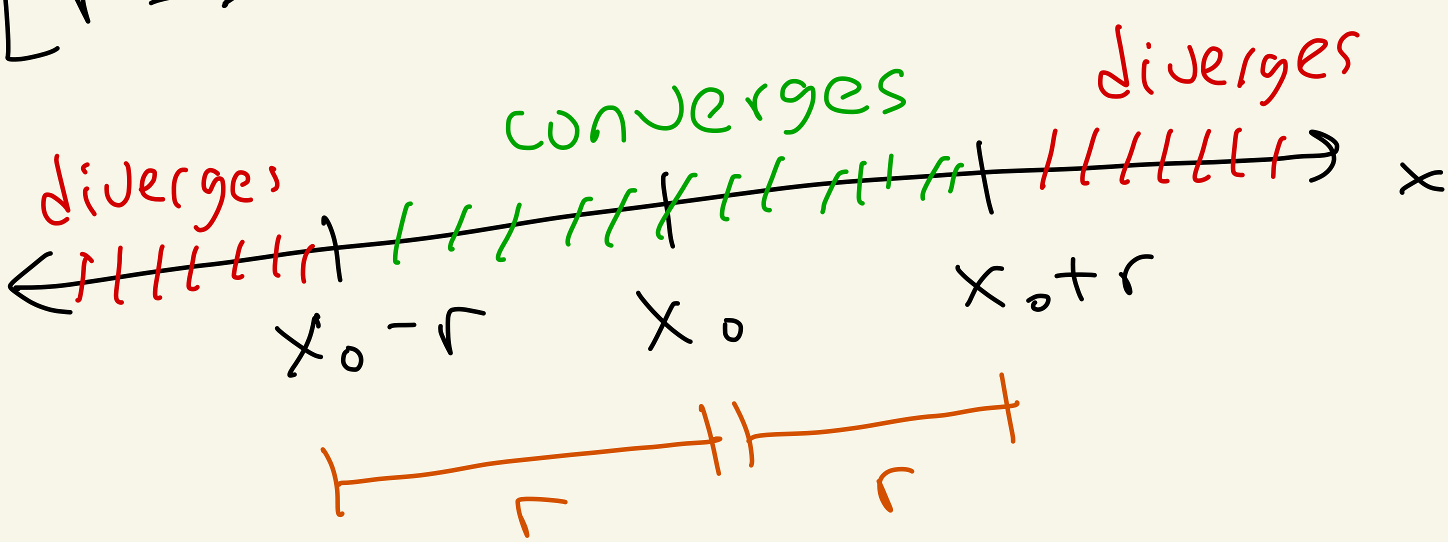


Topic 12 — Power series solutions of ODE

<u>Def</u>: We say that a function $f(x)$ is <u>analytic at x_0</u> if it has a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

centered at x_0 with a radius of convergence $r > 0$
[$r = \infty$ is allowed]

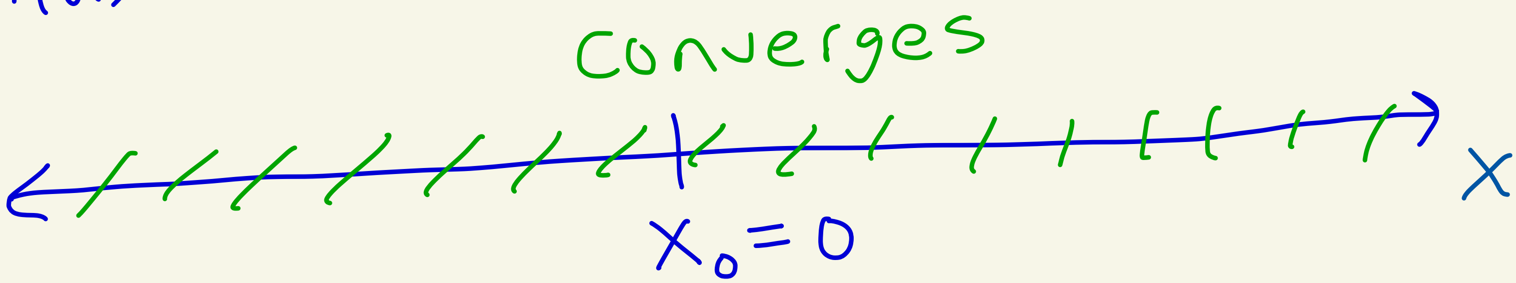


Ex: $f(x) = e^x$ is analytic at $x_0 = 0$ because

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

has radius of convergence $r = \infty$.



Ex: $g(x) = x^2$ is analytic at $x_0 = 2$ because

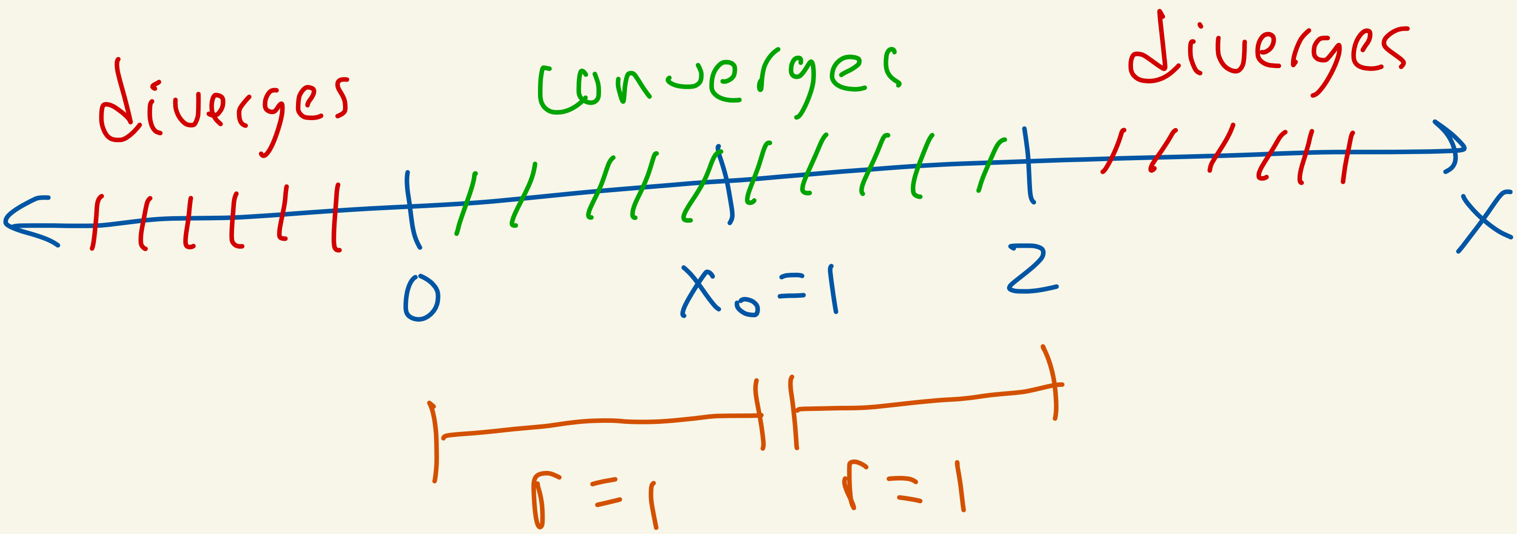
$$x^2 = 4 + 4(x-2) + (x-2)^2$$ last week

has radius of convergence $r = \infty$

Ex: $h(x) = \dfrac{1}{x^2}$ is analytic at $x_0 = 1$ because

$$\frac{1}{x^2} = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \dots$$

has radius of convergence $r = 1$.



Facts:

- polynomials are analytic at every x_0
- e^x, $\sin(x)$, $\cos(x)$ are analytic at every x_0
- $\tan(x)$, $\sec(x)$, $\csc(x)$ are analytic at any x_0 except at their asymptotes
- rational functions (ratios of polynomials) are analytic at every x_0 except possibly where the denominator is zero.

Ex:

$$\left.\begin{array}{l} x^2+5x-2 \\ e^x \\ \sin(x) \\ \cos(x) \end{array}\right\}$$ all analytic for any x_0

Ex: $\underbrace{\dfrac{x}{x^2-1}}_{\text{rational function}}$ is analytic at all x_0 except $\underbrace{x_0 = 1, -1}_{\text{when } x^2-1=0}$

Main Theorem

Consider either of the initial value problems:

$$\boxed{\begin{gathered} y' + a_0(x)y = b(x) \\ y(x_0) = y_0 \end{gathered}}$$

first order

OR

$$\boxed{\begin{gathered} y'' + a_1(x)y' + a_0(x)y = b(x) \\ y'(x_0) = y_0', \ y(x_0) = y_0 \end{gathered}}$$

second order

In either case, if the $a_i(x)$ are analytic at x_0 and $b(x)$ are analytic at x_0 then there is a unique power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

centered at x_0.

Furthermore, the radius of convergence $r > 0$ for the power series of $y(x)$ is at least the smallest radius of convergence from amongst power series of the $a_i(x)$ and $b(x)$.