Math 2150-02
4/23/25

HW 12

(3) Find the first <u>four</u> non-zero terms of the power series solution to

$$\boxed{\begin{gathered} xy'' + x^2y' - 2y = 0 \\ y'(1) = 1, \quad y(1) = 1 \end{gathered}}$$

What's the radius of convergence?

Here $\boxed{x_0 = 1}$

Divide by x to get:

$$y'' + xy' - \frac{2}{x}y = 0$$

<u>coefficients</u>

$x = 1 + (x-1) + 0(x-1)^2 + \ldots \leftarrow$ poly. so $r = \infty$

$-\frac{2}{x} \leftarrow$ we did this in class, $r = 1$

$0 = 0 + 0(x-1) + 0(x-1)^2 + \dots \in$ poly. so
$r = \infty$

So our solution will have $r = 1$

Let's find the solution.

<u>Have:</u> $y(1) = 1$
$y'(1) = 1$

<u>Know:</u> $y'' = -xy' + \frac{2}{x} y$

$$y''(1) = -(1)[y'(1)] + \frac{2}{1}[y(1)]$$

$$= -1 \cdot [1] + 2[1] = 1$$

$$y''(1) = 1$$

Now differentiate $y'' = -xy' + \frac{2}{x} y$
to get

$$y''' = -y' - xy'' - \frac{2}{x^2}y + \frac{2}{x}y'$$

$$y'''(1) = -\left[y'(1)\right] - (1)\left[y''(1)\right] - \frac{2}{(1)^2}\left[y(1)\right] + \frac{2}{(1)}\left[y'(1)\right]$$

$$= -1 - (1)(1) - 2(1) + 2(1)$$

$$= -2$$

So, $y'''(1) = -2$

So,

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \cdots$$

$$= 1 + 1\cdot(x-1) + \frac{1}{2}(x-1)^2 + \frac{-2}{6}(x-1)^3 + \cdots$$

$$y(x) = 1 + (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 + \cdots$$

with radius of convergence $r=1$.

HW 11
1(f)

Find the power series for $f(x) = \frac{1}{x}$ centered at $x_0 = 1$. What is the radius of convergence.

Hint: Use

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

that has radius of convergence $r=1$

If we differentiate

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

we get

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

with radius of convergence $r = 1$

$\boxed{\text{HW 10} \\ 1(e)}$

Given the solution $y_1 = x^{-4}$ for

$$x^2 y'' - 20y = 0$$

on $I = (0, \infty)$, find another linearly independent solution y_2. What's the general solution?

Divide by x^2 to get

$$\underbrace{y'' - \frac{20}{x^2}y = 0}$$

the coefficient of y' is $a_1(x) = 0$

So,

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)dx}}{y_1^2} dx$$

$$= x^{-4} \int \frac{e^{-\int 0 dx}}{(x^{-4})^2} dx$$

$$= x^{-4} \int \frac{e^{-0}}{x^{-8}} dx$$

$\boxed{1}$

$$= x^{-4} \int x^8 dx$$

$$= x^{-4} \cdot \left(\frac{x^9}{9} \right)$$

$$= \frac{1}{9} x^5$$

So, $\boxed{y_2 = \frac{1}{9} x^5}$

Thus, the general solution to

$$x^2 y'' - 20y = 0$$

is

$$y_h = c_1 y_1 + c_2 y_2 = c_1 x^{-4} + c_2 \left(\frac{1}{9} x^5 \right)$$

HW 9
1(c)

Solve $y'' - 9y = \frac{9x}{e^{3x}}$

Use variation of parameters to find y_p

Step 1: Solve the homogeneous equation

$$y'' - 9y = 0$$

The character equation is

$$r^2 - 9 = 0$$

$$(r-3)(r+3) = 0$$

$$r = 3, -3$$

So, $y_h = c_1 e^{3x} + c_2 e^{-3x}$.

Step 2: Find $y_p = v_1 y_1 + v_2 y_2$ for

$$y'' - 9y = \boxed{\frac{9x}{e^{3x}}} \leftarrow b(x)$$

where $y_1 = e^{3x}$, $y_2 = e^{-3x}$

and

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1, y_2)} dx, \quad v_2 = \int \frac{y_1 b(x)}{W(y_1, y_2)} dx$$

We have

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}$$

$$= (e^{3x})(-3e^{-3x}) - (e^{-3x})(3e^{3x})$$

$$= -3 - 3 = -6$$

$$e^{3x}e^{-3x} = e^{3x-3x} = e^{0} = 1$$

or

$$e^{3x}e^{-3x} = e^{3x} \cdot \frac{1}{e^{3x}} = 1$$

So,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)}\,dx = \int \frac{-e^{-3x} \cdot \left(\frac{9x}{e^{3x}}\right)}{-6}\,dx$$

$$= \frac{9}{6} \int \frac{e^{-3x} \cdot x e^{-3x}}{1}\,dx$$

$$= \frac{9}{6} \int x e^{-6x}\,dx$$

LIATE

$$u = x \qquad du = dx$$

$$dv = e^{-6x}\,dx \qquad v = -\frac{1}{6}e^{-6x}$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$= \frac{9}{6}\left[x\left(-\frac{1}{6}e^{-6x}\right) - \int\left(-\frac{1}{6}e^{-6x}\right)dx\right]$$

$$= -\frac{9}{36}xe^{-6x} + \frac{9}{36}\int e^{-6x}\,dx$$

$$= -\frac{9}{36}xe^{-6x} + \frac{9}{36}\left[\frac{1}{-6}e^{-6x}\right]$$

$$= \boxed{-\frac{9}{36}xe^{-6x} - \frac{9}{216}e^{-6x}} \quad v_1$$

And

$$v_2 = \int \frac{y_1\,b(x)}{W(y_1,y_2)}\,dx = \int \frac{\cancel{e^{3x}}\left(\frac{9x}{\cancel{e^{3x}}}\right)}{-6}\,dx$$

$$= -\frac{1}{6}\int 9x\,dx = -\frac{1}{6}\left(9\frac{x^2}{2}\right)$$

$$= \boxed{-\frac{3}{4}x^2}$$

Thus,

$$y_p = v_1 y_1 + v_2 y_2$$
$$= \left(-\frac{9}{36}xe^{-6x} - \frac{9}{216}e^{-6x}\right)e^{3x}$$
$$+ \left(-\frac{3}{4}x^2\right)e^{-3x}$$

The general solution to $y'' - 9y = \frac{9x}{e^{3x}}$

is $y = y_h + y_p$

$$= c_1 e^{3x} + c_2 e^{-3x}$$
$$+ \left(-\frac{9}{36}xe^{-6x} - \frac{9}{216}e^{-6x}\right)e^{3x}$$
$$- \frac{3}{4}x^2 e^{-3x}$$

HW 8
1(d)

Find y_p to $y'' + 3y = xe^{3x}$ using undetermined coefficients given $y_h = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$

Guess: $y_p = (Ax+B)e^{3x} = Axe^{3x} + Be^{3x}$

$$y_p' = Ae^{3x} + 3Axe^{3x} + 3Be^{3x}$$

$$y_p'' = 3Ae^{3x} + 3Ae^{3x} + 9Axe^{3x} + 9Be^{3x}$$

Plug these into $y'' + 3y = xe^{3x}$.

Get:

$$\overbrace{(3Ae^{3x} + 3Ae^{3x} + 9Axe^{3x} + 9Be^{3x})}^{y_p''} + 3\underbrace{(Axe^{3x} + Be^{3x})}_{} = xe^{3x}$$

y_p

Get:

$$6Ae^{3x} + 9Axe^{3x} + 9Be^{3x} + 3Axe^{3x} + 3Be^{3x} = xe^{3x}$$

Regroup:

$$\underbrace{(6A+12B)}_{0}e^{3x} + \underbrace{(12A)}_{1}xe^{3x} = xe^{3x}$$

So, $\boxed{\begin{aligned} 6A+12B &= 0 \\ 12A &= 1 \end{aligned}} \leftarrow A = 1/12$

So, $6\left(\frac{1}{12}\right) + 12B = 0$

$$12B = -\frac{1}{2}$$

$$B = -\frac{1}{24}$$

Thus,

$$\boxed{y_p = (Ax+B)e^{3x} = \left(\frac{1}{12}x - \frac{1}{24}\right)e^{3x}}$$