$\boxed{\begin{array}{l}\text{Math 2150} \\ \text{8-26-24}\end{array}}$

Ex:

$$y'' - y = 0$$

ODE
$y = y(x)$

2nd order

Find a function to plug in for y to solve the eqn.

Let $y = e^x$.
Then $y' = e^x$, $y'' = e^x$.

So,
$$y'' - y = e^x - e^x = 0$$
So $y = e^x$ solves the ODE.

Def: An ODE is called **linear** if it is of the form

$$\underbrace{a_n(x)}\,y^{(n)} + \underbrace{a_{n-1}(x)}\,y^{(n-1)} + \dots + \underbrace{a_1(x)}\,y' + \underbrace{a_0(x)}\,y = \underbrace{b(x)}$$

that is, the coefficients of $y^{(k)}$ terms only have x's and constants in them

Ex:

$$\underbrace{x^2}_{a_2(x) = x^2} y'' \underbrace{- 3}_{a_1(x) = -3} y' + \underbrace{2\ln(x)}_{a_0(x) = 2\ln(x)} y = \underbrace{3x}_{b(x)}$$

is a linear ODE.

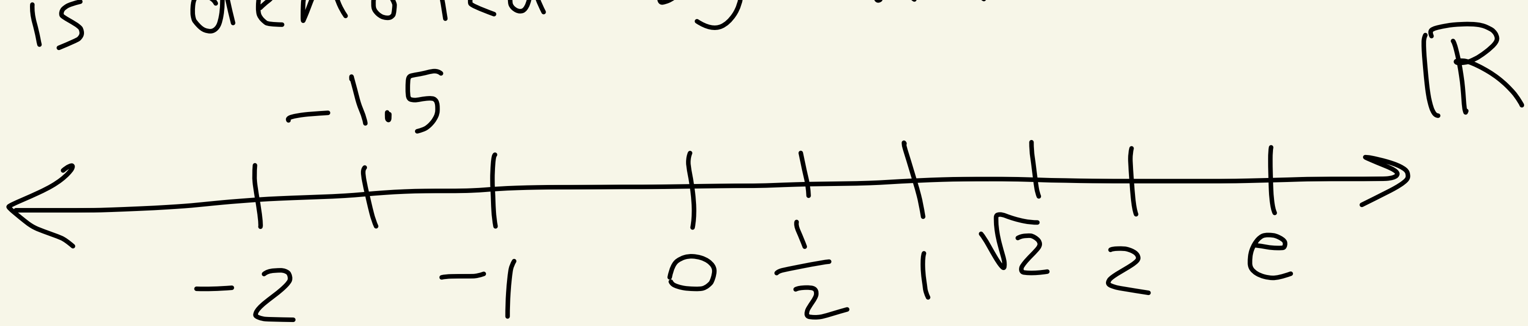
Ex: $y' - 13x^3 y = 10$ linear

Ex: $y^2 y'' + 10xy = x^3$ not linear

Ex: $\underbrace{x^3}_{\checkmark} y^{(10)} \underbrace{- 5x\sin(x)}_{\checkmark} y = \underbrace{\sin(y)}_{\times}$

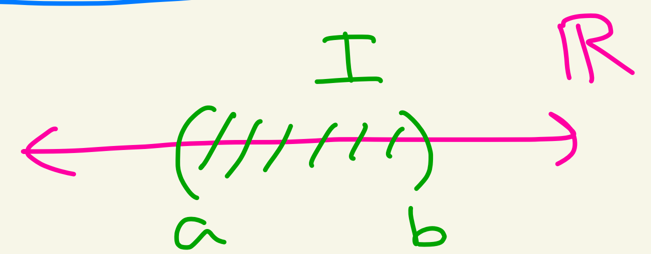
not linear

Def: The real number line is denoted by $\mathbb{R}$



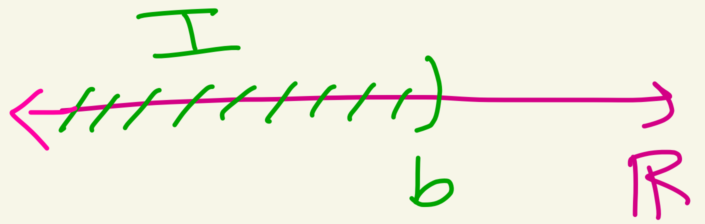
<u>Def:</u> An <u>open interval</u> I is an interval of the form

$I = (a, b)$ $\longleftrightarrow$



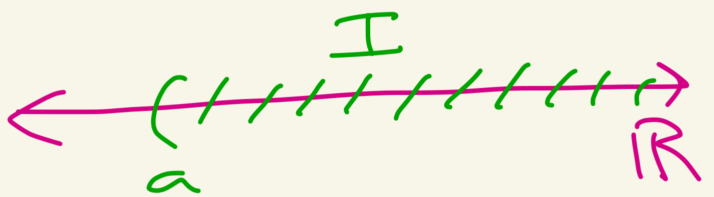
$a < x < b$

or $I = (-\infty, b)$ $\longleftrightarrow$



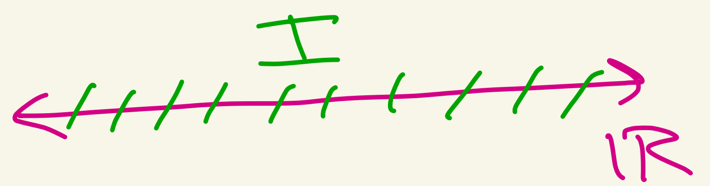
$-\infty < x < b$

or $I = (a, \infty)$ $\longleftrightarrow$



$a < x < \infty$

or $I = (-\infty, \infty)$ $\longleftrightarrow$



$-\infty < x < \infty$

Def: A real-valued function f is a solution to an n-th order ODE on an open interval I if

① $f, f', f'', \ldots, f^{(n)}$ exist on I

and ② when you plug f and its derivatives into the ODE it solves the equation for all x in I.

In addition, sometimes one is given what $f(x_0), f'(x_0), \ldots, f^{(n-1)}(x_0)$ must equal for some x_0 in I. This extra condition turns the ODE into an initial-value problem

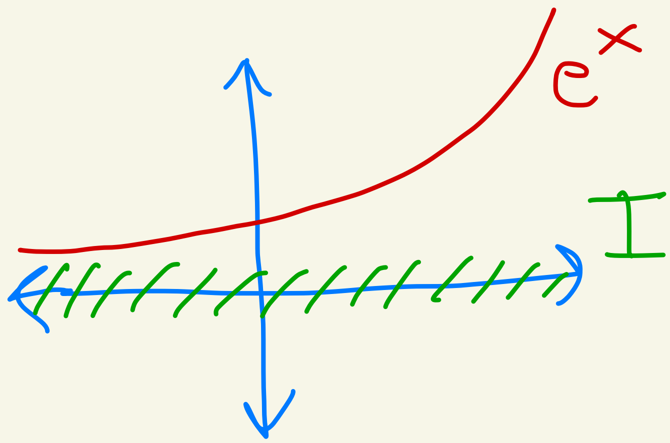
Ex: Let's show that $f(x) = e^x$ is a solution to

$$y' - y = 0$$

order 1 linear ODE

on the interval $I = (-\infty, \infty)$

① $f(x) = e^x$
$f'(x) = e^x$



Both f and f' exist on I.

② If you plug in $y = f = e^x$ and $y' = f' = e^x$ then

$$y' - y = e^x - e^x = 0$$

works for all x in I

So, f solves $y' - y = 0$ on I.

Ex: Consider the ODE

$$y'' - 4y = 0$$

on $I = (-\infty, \infty)$

Let $f(x) = ce^{2x}$ where c is any constant.

Let's show f solves the ODE on $I = (-\infty, \infty)$.

① $f(x) = ce^{2x}$
$f'(x) = 2ce^{2x}$
$f''(x) = 4ce^{2x}$

these are all defined on $I = (-\infty, \infty)$

② Plugging these into the ODE we get $(y = f, y' = f', y'' = f'')$

$$y'' - 4y = 4ce^{2x} - 4(ce^{2x}) = 0$$

So, $y = ce^{2x}$ solves $y'' - 4y = 0$ on $I = (-\infty, \infty)$.

<u>Note:</u> c can be any number so we just found an infinite # of solutions to $y'' - 4y = 0$ on $I = (-\infty, \infty)$.

Some are: $\underbrace{2e^{2x}}_{c=2}$, $\underbrace{10e^{2x}}_{c=10}$, $\underbrace{-\frac{1}{2}e^{2x}}_{c=-\frac{1}{2}}$, ...

<u>Ex:</u> Consider the initial value problem

$$y'' - 4y = 0$$

$$y(0) = 5, \; y'(0) = 10 \leftarrow \boxed{\text{initial conditions}}$$

We want to solve this on $I=(-\infty,\infty)$

We know $y=ce^{2x}$ solves

$$y''-4y=0.$$

Can we make it also satisfy $y(0)=5$, $y'(0)=10$?

$$y=ce^{2x}$$
$$y'=2ce^{2x}$$

$y(0)=5$ becomes

$$5=ce^{2(0)}$$

So need

$$5=c\underbrace{e^{0}}_{1}$$
$$5=c$$

$y'(0)=10$ becomes

$$10=2ce^{2(0)}$$
$$10=2c$$
$$5=c$$

So, $y = \underbrace{5}_{c=5} e^{2x}$ will solve

$$\boxed{\begin{aligned} &y'' - 4y = 0 \\ &y(0) = 5,\ y'(0) = 10 \end{aligned}}$$

on $I = (-\infty, \infty)$

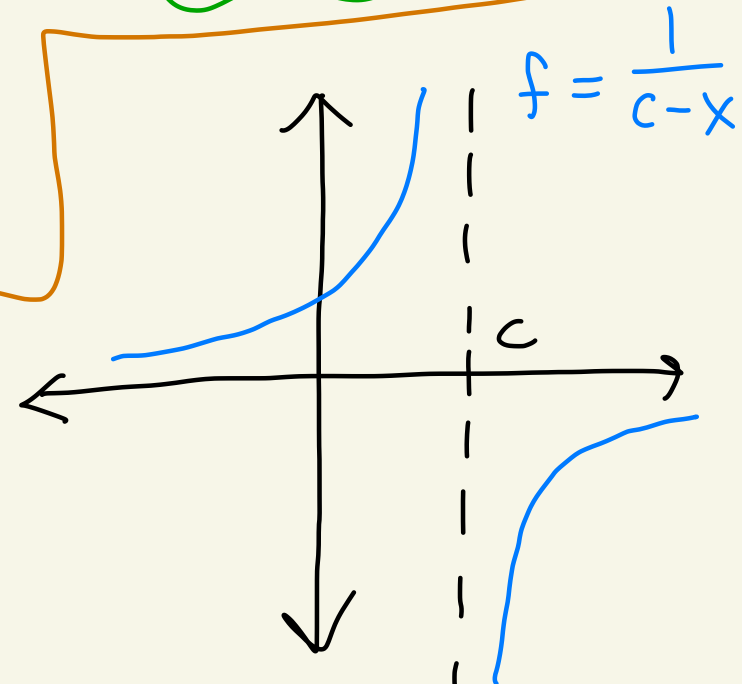
Ex: Consider the non-linear ODE

$$\boxed{\frac{dy}{dx} = y^2}$$

same as: $y' = y^2$

Let $f(x) = \dfrac{1}{c-x}$

where c is a constant.



Then,

$$f'(x) = -(c-x)^{-2} \cdot (-1) = \frac{1}{(c-x)^2}$$

$$\boxed{f(x) = (c-x)^{-1}}$$

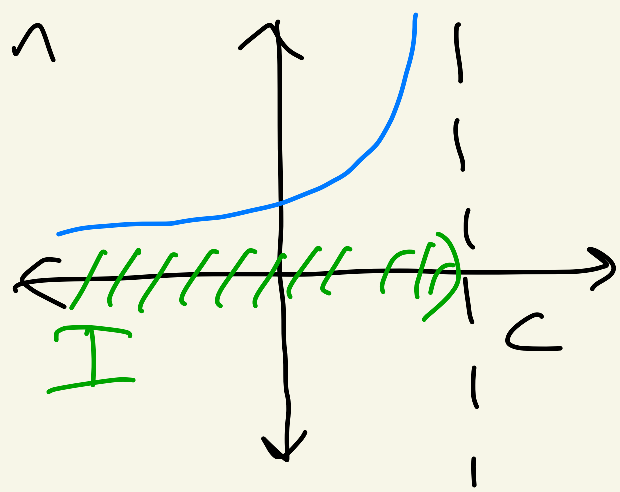
EQUAL

Then, $(f(x))^2 = \left(\dfrac{1}{c-x}\right)^2 = \dfrac{1}{(c-x)^2}$

So, $f'(x) = (f(x))^2$.

Thus, f solves $\frac{dy}{dx} = y^2$.

You can say that $f(x) = \frac{1}{c-x}$ solves $y' = y^2$ on $I = (-\infty, c)$



OR you could say that f solves $y' = y^2$ on $I = (c, \infty)$.