Math 2150
9/11/24

Ex: Consider the ODE

$$\underbrace{2xy}_{M(x,y)} + \underbrace{(x^2-1)}_{N(x,y)}\, y' = 0$$

Let

$$f(x,y) = x^2 y - y$$

We will see how to find f later

Then,

$$\frac{\partial f}{\partial x} = 2xy = M(x,y)$$

$$\frac{\partial f}{\partial y} = x^2 - 1 = N(x,y)$$

So, an implicit solution to

$$2xy + (x^2-1)y' = 0$$

is given by

$$x^2y - y = c$$

where c is any constant.

$f(x,y) = c$

Check #1:

Suppose $x^2y - y = c$ where y is a function of x. Differentiate both sides with respect to x to get:

$$2xy + x^2y' - y' = 0$$

This is

$$2xy + (x^2-1)y' = 0$$

which is the original equation

We can actually solve our solution $x^2y - y = c$ for y and we get:

$$y = \frac{c}{x^2-1} \leftarrow \boxed{\text{defined when } x \neq \pm 1}$$

Check #2

Does $y = c(x^2-1)^{-1}$ solve $2xy + (x^2-1)y' = 0$?

We need

$$y' = -c(x^2-1)^{-2} \cdot (2x) = \frac{-2xc}{(x^2-1)^2}$$

Let's plug it in!

$$2xy + (x^2-1)y'$$

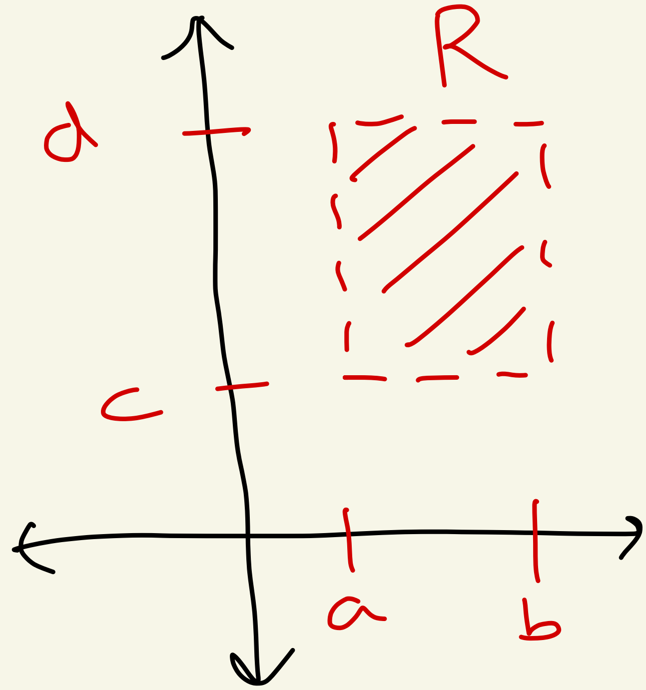
$$= 2x\left(\frac{c}{x^2-1}\right) + (x^2-1)\left(\frac{-2xc}{(x^2-1)^2}\right)$$

$$= \frac{2xc}{x^2-1} - \frac{2xc}{x^2-1} = 0$$

Yes! It works.

When will such an f exist?

Theorem: Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in some rectangle R defined by $a<x<b$ and $c<y<d$.



Then

$$M(x,y)+N(x,y)y'=0$$

will be exact if and only if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Note a, b, c, d can be $\pm\infty$

proof: See online notes if interested

Ex: Look at the previous equation:

$$\underbrace{2xy}_{M} + \underbrace{(x^2-1)}_{N}\, y' = 0$$

Let $M(x,y) = 2xy$, $N(x,y) = x^2 - 1$

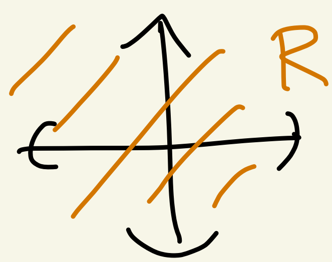
M and N are continuous everywhere.

$$\frac{\partial M}{\partial x} = 2y \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial y} = 0$$

continuous everywhere

Here R is the entire xy-plane.



Since $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$

the equation $2xy + (x^2-1)y' = 0$ will be exact by the theorem.

<u>Ex:</u> Let's see how to find f for an exact equation.

Consider

$$\underbrace{2xy}_{M} + \underbrace{(x^2-1)}_{N}y' = 0$$

We want $f(x,y)$ where

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2xy \quad ① \\ \frac{\partial f}{\partial y} &= x^2-1 \quad ② \end{aligned}}$$

$\boxed{\partial f/\partial x = M}$

$\boxed{\partial f/\partial y = N}$

Start with 1:

$$\frac{\partial f}{\partial x} = 2xy$$

Integrate both sides with respect to x to get:

$$f(x,y) = x^2 y + \underbrace{g(y)}_{\text{constant of integration}}$$

Differentiate with respect to y:

$$\frac{\partial f}{\partial y} = x^2 + g'(y)$$

Plug in 2 $\frac{\partial f}{\partial y} = x^2 - 1$ to get

$$x^2 - 1 = x^2 + g'(y)$$

So,

$$-1 = g'(y)$$

Thus,

$$g(y) = -y$$

We don't need $-y + c$ because we will set f equal to a constant

Therefore,

$$f(x,y) = x^2 y + g(y)$$
$$= x^2 y - y.$$

So,

$$x^2 y - y = c$$

$\boxed{f(x,y) = c}$

solves

$$2xy + (x^2 - 1)\,y' = 0.$$

HW EXAMPLE FROM SEPARABLE EQUATIONS

HW #4
1 (c)

Find a solution to

$$\frac{dy}{dx} = \frac{-x}{y}$$

If we can solve for y in our solution, then do so and give an interval where the solution exists.

We get

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

$\downarrow$

$$\boxed{y\,dy = -x\,dx}$$

$$\downarrow$$

$$\boxed{\int y\,dy = -\int x\,dx}$$

$$\downarrow$$

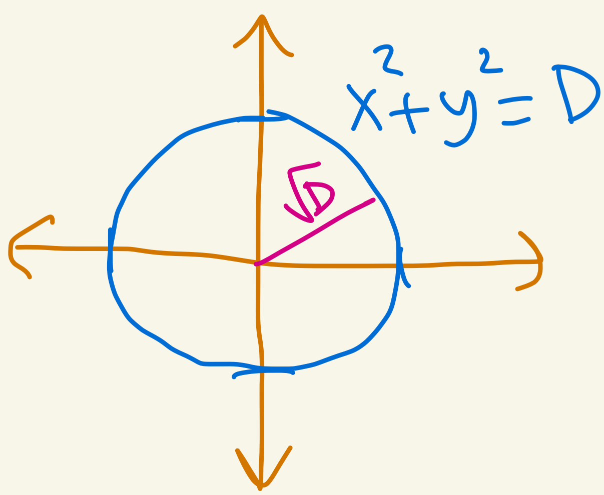
$$\boxed{\frac{y^2}{2} = -\frac{x^2}{2} + C}$$

$$\downarrow$$

$$\boxed{y^2 = -x^2 + D} \longleftarrow \boxed{D = 2C}$$

$$\downarrow$$

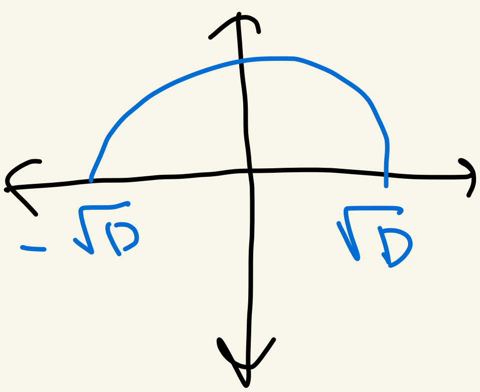
$$\boxed{x^2 + y^2 = D}$$



Can solve for y and you get two solutions:

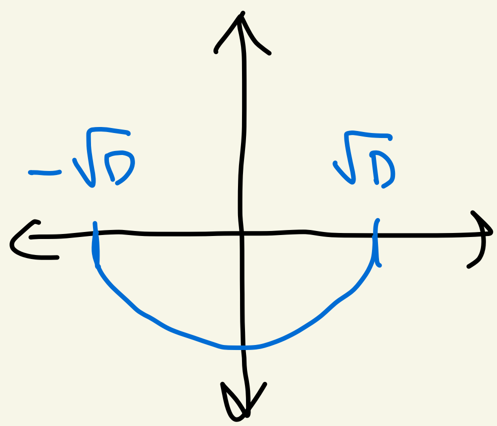
Solution 1

$$y = \sqrt{D - x^2}$$



Solution 2

$$y = -\sqrt{D - x^2}$$



Both are defined on $I = (-\sqrt{D}, \sqrt{D})$

HW 4 #1(d) Same question for

$$\boxed{\begin{gathered}\frac{dy}{dx} = \frac{-x}{y} \\ y(4) = 3\end{gathered}}$$

We already know $\frac{dy}{dx} = \frac{-x}{y}$ is solved by $x^2+y^2=D$.

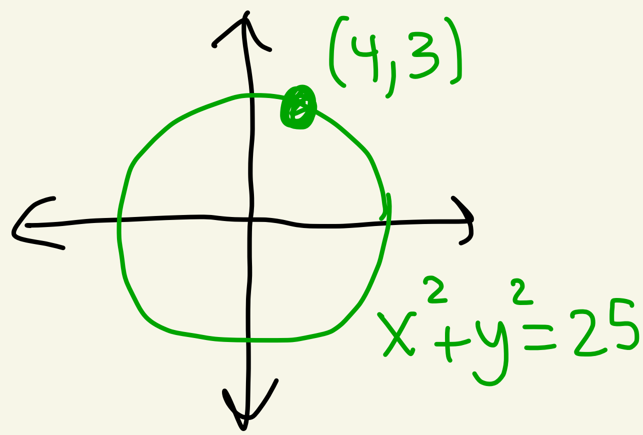
Plug in $y(4)=3$, ie $x=4, y=3$

to get: $16+9=D$

So, $D=25$.

Thus, we get the solution

$$x^2+y^2=25$$



Take the top half to get

$$y=\sqrt{25-x^2}$$

This will solve $\frac{dy}{dx}=\frac{-x}{y}$ with $y(4)=3$.