Math 2150
9/16/24

Topic 6 – Theory of second order linear ODEs

Up to this point we've been learning techniques to solve first order ODEs (linear, separable, exact). There's a bunch more methods if you look in books.

Now we will look at second order equations but only linear ones.

We will consider second order linear ODEs of the form:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

where $a_2(x), a_1(x), a_0(x), b(x)$ are continuous on some interval I.

For now we will assume $a_2(x) \neq 0$ for all x in I.

Then we can divide it out and make the y'' coefficient equal to 1.

We will also consider initial conditions

$$y'(x_0) = y_0' \text{ and } y(x_0) = y_0$$

at some point x_0 in I.

$$\underbrace{y'(x_0)}_{\text{derivate}} = \underbrace{y_0^{\,\prime}}_{\text{a number}}$$

prime not derivative

Ex: Consider

$$y'' - 7y' + 10y = 24e^{x}$$

on $I = (-\infty, \infty)$.

Let

$$f(x) = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

where c_1, c_2 are constants.

Note f is defined on $I = (-\infty, \infty)$.

Let's show that f solves

the ODE.

We have

$$f(x) = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$$

$$f'(x) = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^{x}$$

$$f''(x) = 4c_1 e^{2x} + 25c_2 e^{5x} + 6e^{x}$$

Plug f into the left-side of the ODE to get:

$$\begin{aligned} y'' - 7y' + 10y &= f'' - 7f' + 10f \\ &= \left(4c_1 e^{2x} + 25c_2 e^{5x} + 6e^{x}\right) \\ &\quad - 7\left(2c_1 e^{2x} + 5c_2 e^{5x} + 6e^{x}\right) \\ &\quad + 10\left(c_1 e^{2x} + c_2 e^{5x} + 6e^{x}\right) \end{aligned}$$

$$= 4c_1e^{2x} + 25c_2e^{5x} + 6e^{x}$$
$$-14c_1e^{2x} - 35c_2e^{5x} - 42e^{x}$$
$$+10c_1e^{2x} + 10c_2e^{5x} + 60e^{x}$$

$$= 24e^{x}$$

So, $f(x) = c_1e^{2x} + c_2e^{5x} + 6e^{x}$ solves $y'' - 7y' + 10y = 24e^{x}$ on $I = (-\infty, \infty)$.

<u>Ex:</u> Use the above to solve:

$$\boxed{\begin{gathered} y'' - 7y' + 10y = 24e^{x} \\ y'(0) = 6,\ y(0) = 0 \end{gathered}}$$

on $I = (-\infty, \infty)$.

We Know $f(x) = c_1 e^{2x} + c_2 e^{5x} + 6e^x$

solves $y'' - 7y' + 10y = 24e^x$.

Let's see if we can make

it solve $\boxed{y'(0) = 6,\ y(0) = 0.}$

Recall $f'(x) = 2c_1 e^{2x} + 5c_2 e^{5x} + 6e^x$

Want

$$\boxed{\begin{aligned} f(0) &= 0 \\ f'(0) &= 6 \end{aligned}}$$

This is

$$\boxed{\begin{aligned} c_1 e^{2(0)} + c_2 e^{5(0)} + 6e^0 &= 0 \\ 2c_1 e^{2(0)} + 5c_2 e^{5(0)} + 6e^0 &= 6 \end{aligned}}$$

$e^0 = 1$

$$\boxed{\begin{aligned} c_1 + c_2 &= -6 \quad (1) \\ 2c_1 + 5c_2 &= 0 \quad (2) \end{aligned}}$$

(1) gives $c_1 = -6 - c_2$

Plug into (2) to get

$$2(\underbrace{-6 - c_2}_{c_1}) + 5c_2 = 0$$

Giving

$$3c_2 = 12$$

So, $c_2 = 4$.

And, $c_1 = -6 - c_2 = -6 - 4 = -10$.

Thus,

$$f(x) = \underbrace{-10}_{c_1}e^{2x} + \underbrace{4}_{c_2}e^{5x} + 6e^{x}$$

Solves

$$\boxed{\begin{array}{l} y'' - 7y' + 10y = 24e^{x} \\ y'(0) = 6, \; y(0) = 0 \end{array}}$$

Theorem

Let I be an interval.
Let $a_2(x)$, $a_1(x)$, $a_0(x)$, $b(x)$ be continuous on I and $a_2(x) \neq 0$ for all x in I.

Let x_0 be in I.

Then,

$$\boxed{\begin{array}{l} a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x) \\ y'(x_0) = y_0', \; y(x_0) = y_0 \end{array}}$$

will have a unique solution on I.

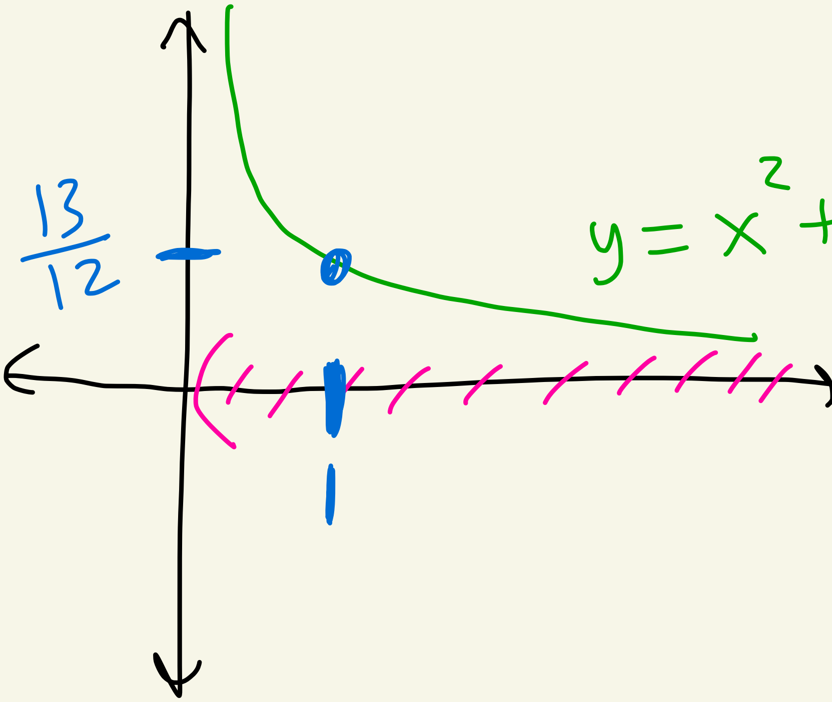
Ex: Consider

$$\boxed{\begin{gathered} x^2y'' - 4xy' + 6y = \frac{1}{x} \\ y'(1) = \frac{13}{12} \quad y(1) = \frac{23}{12} \end{gathered}}$$

on $I = (0, \infty)$ $\leftarrow$ $\boxed{x > 0}$

Theorem says this has a unique solution. In HW 10 you'll find it. It's

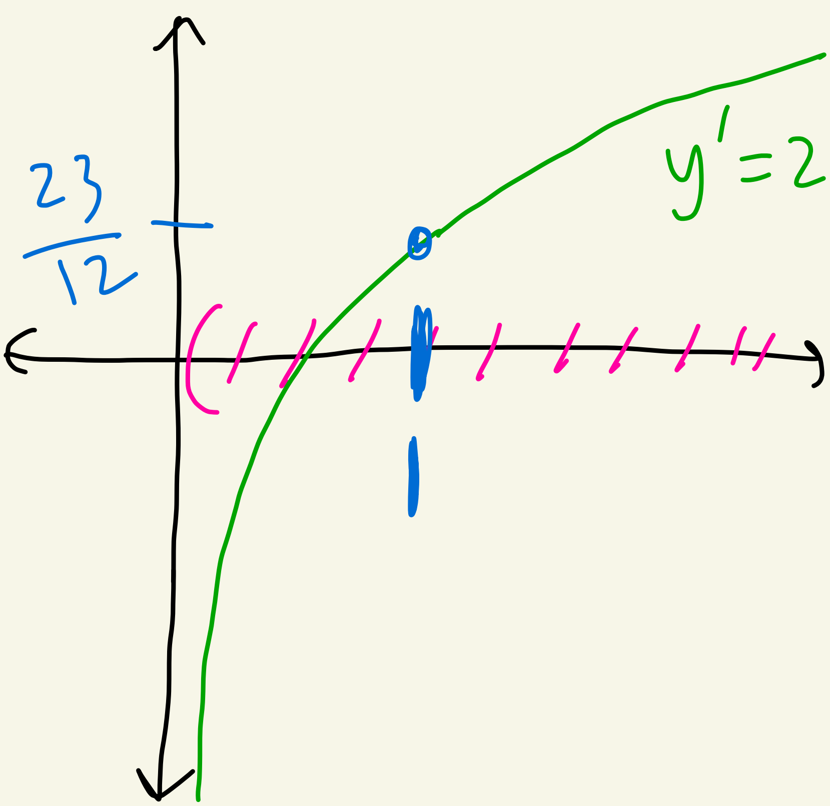
$$y = x^2 + \frac{1}{12x}$$



$y = x^2 + \frac{1}{12x}$

$I = (0, \infty)$

$y(1) = \frac{13}{12}$



$y' = 2x - \frac{1}{12x^2}$

$I = (0, \infty)$

$y'(1) = \frac{23}{12}$

<u>Def:</u> Let I be an interval.
Let f_1, f_2 be functions defined on I.
We say that f_1, f_2 are <u>linearly dependent</u> on I if one of them is a multiple of the other on I. That is, if either

$$f_2(x) = c f_1(x) \text{ for all } x \text{ in } I$$

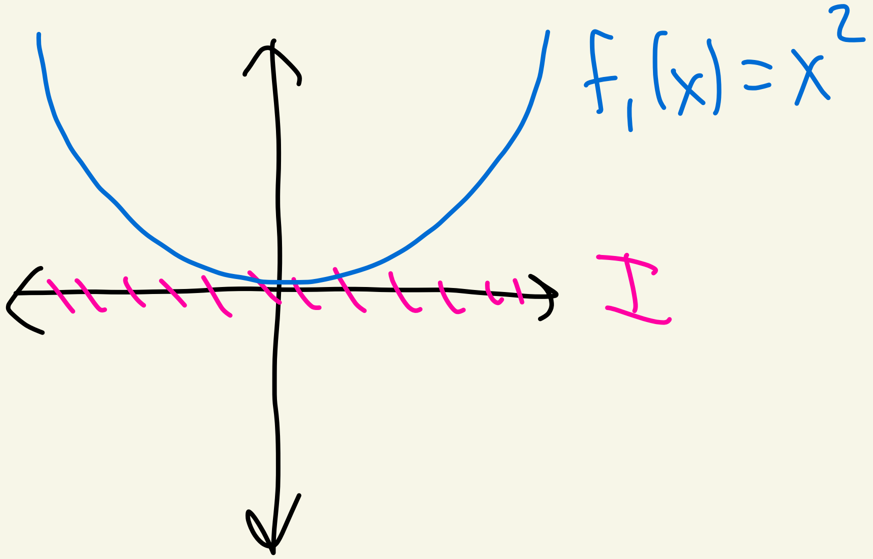
or

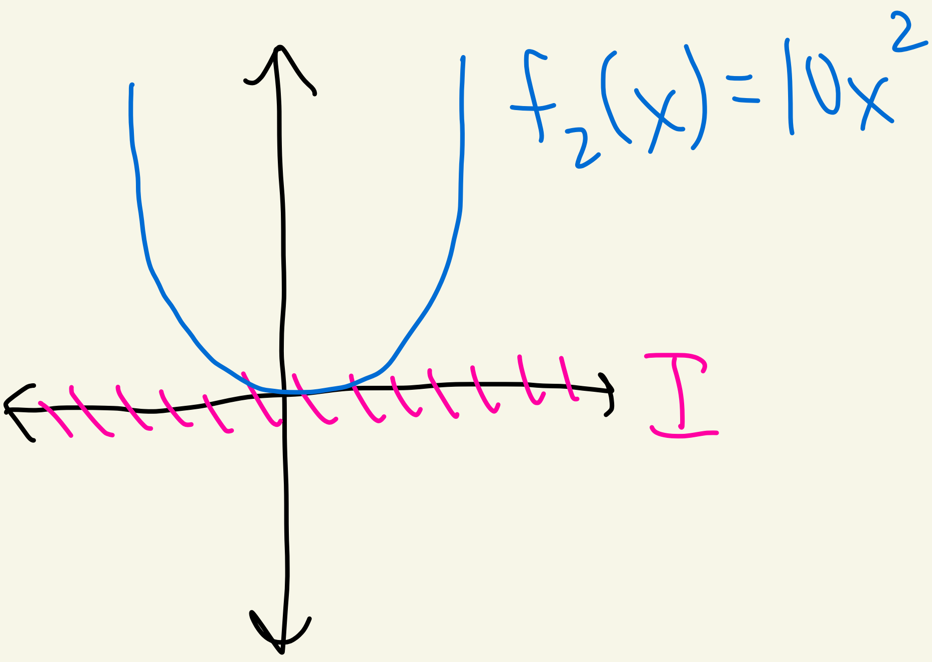
$$f_1(x) = c f_2(x) \text{ for all } x \text{ in } I$$

where c is a constant.
If they aren't linearly dependent they are called <u>linearly independent</u>.

Ex: Let $f_1(x) = x^2$ and $f_2(x) = 10x^2$.

Let $I = (-\infty, \infty)$





f_1 and f_2 are linearly dependent on I because $f_2(x) = 10 f_1(x)$ for all x in I.