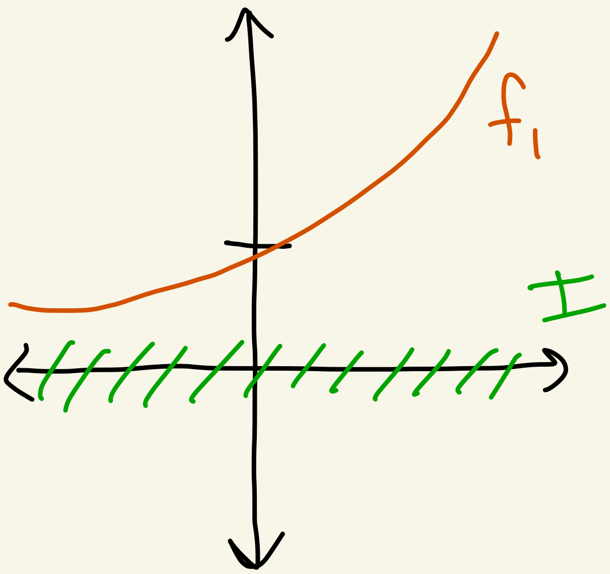
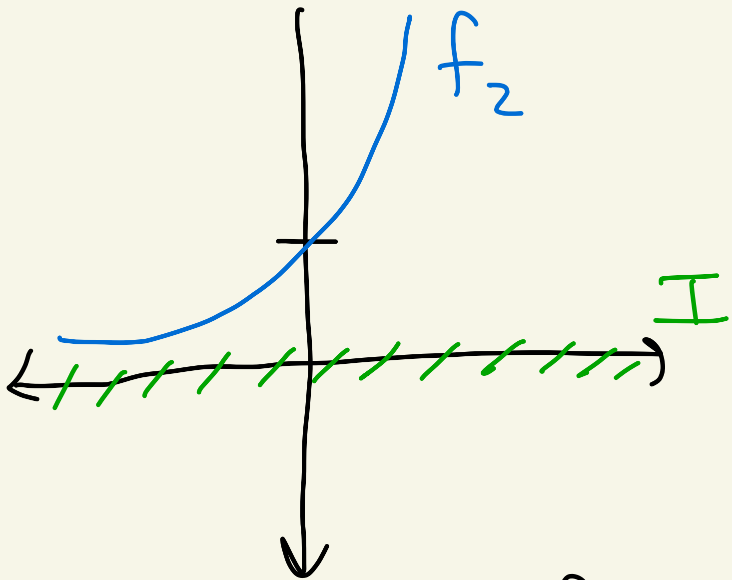
Math 2150
9/18/24

Ex: Let $I = (-\infty, \infty)$.

Let $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$





Let's show that f_1 and f_2 are linearly independent on I.

Why?

Suppose $f_1(x) = c f_2(x)$ on I.

Then $e^{2x} = c e^{5x}$ for all x in I.

Then $x = 0$ gives $1 = c$

And $x = 1$ gives $e^2 = c e^5$

which gives $e^{-3} = c$.

But then $1 = c = \underbrace{e^{-3}}_{\approx 0.049787\ldots}$

This can't happen!

Similarly you can show that there is no c where $f_2(x) = c\, f_1(x)$ on I.

Thus, $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$ are linearly independent on I.

We will now learn another way to check for linear independence on I.

It's a method named after Josef Wronski (1778-1853)

<u>Def:</u> Let f_1 and f_2 be differentiable on an interval I. The <u>Wronskian</u> of f_1 and f_2 is the following determinant

$$W(f_1,f_2) = \underbrace{\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}}_{\text{this is called a determinant}} = f_1 f_2' - f_2 f_1'$$

picture: $\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$

Ex: $f_1(x) = e^{2x}, \quad f_2(x) = e^{5x}$

$f_1'(x) = 2e^{2x}, \quad f_2'(x) = 5e^{5x}$

$$W(f_1, f_2) = \begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix}$$

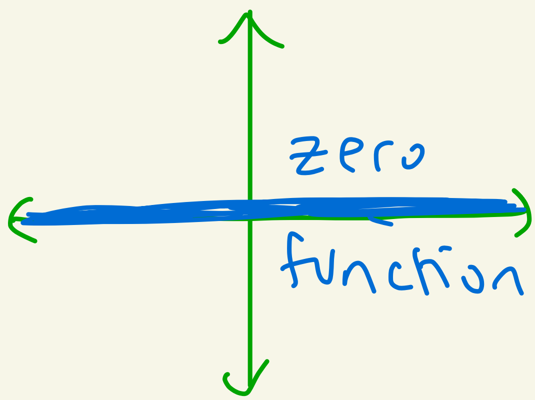
$$= (e^{2x})(5e^{5x}) - (e^{5x})(2e^{2x})$$

$$= 5e^{2x+5x} - 2e^{5x+2x}$$

$$= 3e^{7x}$$

Theorem: Let I be an interval. Let f_1, f_2 be differentiable on I.

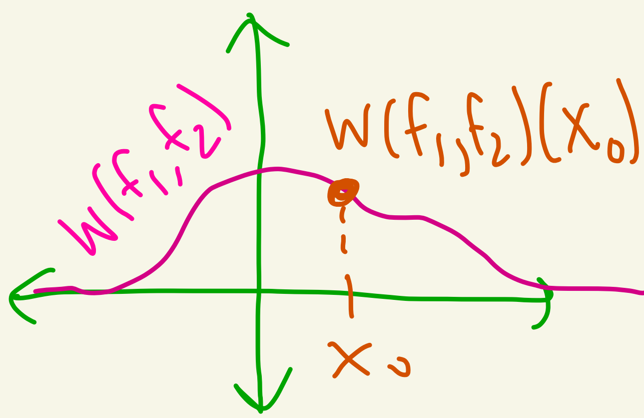
If the Wronskian $W(f_1,f_2)$ is <u>not</u> the zero function on I, then f_1, f_2 are linearly independent on I.



That is, if there exists some point x_0 in I where

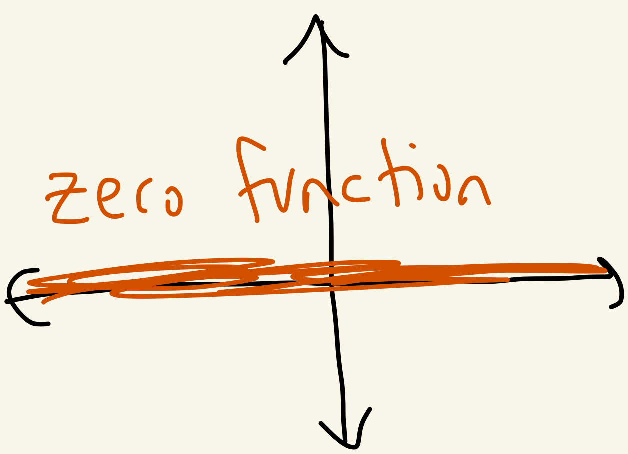
$$W(f_1,f_2)(x_0) \neq 0$$

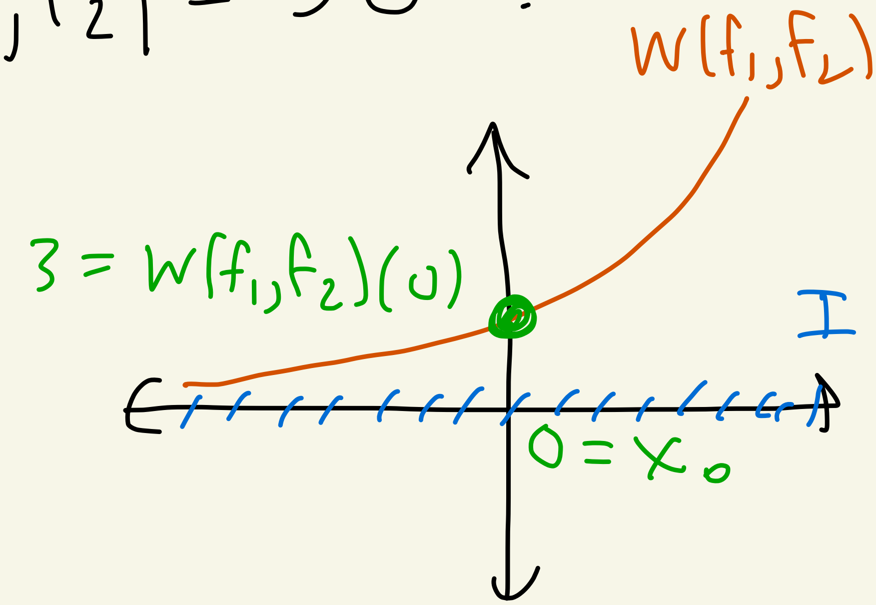
then f_1, f_2 are linearly independent.



Ex: Let's show $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$ are linearly independent on $I = (-\infty, \infty)$ using the Wronskian.

We saw $W(f_1, f_2) = 3e^{7x}$.





The Wronskian $W(f_1, f_2) = 3e^{7x}$ is not the zero function on I. For example at $x_0 = 0$ we get

$$W(f_1, f_2)(0) = 3e^{7(0)} = 3 \cdot e^0 = 3 \neq 0$$

Thus, f_1, f_2 are lin. ind. on I.

Theorem [linear, homogeneous, 2nd order ODEs]

Let I be an interval.
Let $a_2(x), a_1(x), a_0(x)$ be continuous on I. Suppose $a_2(x) \neq 0$ for all x in I.

Consider the homogeneous equation

$$\boxed{a_2(x)y'' + a_1(x)y' + a_0(x)y = 0} \quad (*)$$

homogeneous is this 0

Suppose

- $f_1(x)$ and $f_2(x)$ are linearly independent on I,

and

- $f_1(x)$ and $f_2(x)$ both solve $(*)$

Then every solution to $(*)$ on I is of the form

$$y_h = c_1 f_1(x) + c_2 f_2(x)$$

y_h for homo-geneous

where c_1, c_2 are constants.

Ex: Consider the homogeneous 2nd order linear ODE

$$y'' - 7y' + 10y = 0$$

on $I = (-\infty, \infty)$.

Let $f_1(x) = e^{2x}$ and

$$f_2(x) = e^{5x}.$$

We know that f_1 and f_2 are linearly independent on I.

Let's show they both solve

$$y'' - 7y' + 10y = 0$$

by plugging them in.

We have:

$$f_1(x) = e^{2x} \qquad f_2(x) = e^{5x}$$

$$f_1'(x) = 2e^{2x} \qquad f_2'(x) = 5e^{5x}$$

$$f_1''(x) = 4e^{2x} \qquad f_2''(x) = 25e^{5x}$$

Thus,

$$\begin{aligned} f_1'' - 7f_1' + 10f_1 &= 4e^{2x} - 7(2e^{2x}) + 10(e^{2x}) \\ &= 4e^{2x} - 14e^{2x} + 10e^{2x} \\ &= 0 \end{aligned}$$

So, f_1 solves $y'' - 7y' + 10y = 0$ on I.

Also,

$$f_2'' - 7f_2' + 10f_2$$
$$= 25e^{5x} - 7(5e^{5x}) + 10e^{5x}$$
$$= 25e^{5x} - 35e^{5x} + 10e^{5x}$$
$$= 0$$

So, f_2 solves $y'' - 7y' + 10y = 0$ on I.

Since f_1, f_2 are lin. ind. on I and they both solve the homogeneous equation we know any solution to $y'' - 7y' + 10y = 0$ is of the form

y_h

$$y_h = c_1 f_1 + c_2 f_2 = \boxed{c_1 e^{2x} + c_2 e^{5x}}$$

Theorem: (General linear 2nd order)

Let I be an interval. Let $a_2(x), a_1(x), a_0(x), b(x)$ be continuous on I and $a_2(x) \neq 0$ for all x in I.

Consider

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

- Suppose f_1 and f_2 are linearly independent on I and both solve the homogeneous equation
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$
- Suppose y_p is a particular solution to
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$
on I.

Then, every solution to

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$$

on I is of the form

$$y = \underbrace{c_1 f_1(x) + c_2 f_2(x)}_{y_h} + \underbrace{y_p(x)}_{y_p}$$

where c_1, c_2 are constants.

Ex: Consider

$$y'' - 7y' + 10y = 24e^x$$

on $I = (-\infty, \infty)$.

We know $f_1(x) = e^{2x}$ and $f_2(x) = e^{5x}$ are lin. ind. solutions to the homogeneous equation $y'' - 7y' + 10y = 0$

on I.

Also if $y_p = 6e^x$.

This is a particular solution to $y'' - 7y' + 10y = 24e^x$

because

$$y_p'' - 7y_p' + 10y_p$$
$$= 6e^x - 7(6e^x) + 10(6e^x)$$
$$= 24e^x$$

Thus, the theorem tells that every solution to $y'' - 7y' + 10y = 24e^x$ on I is of the form

$$y = y_h + y_p = \underbrace{c_1 e^{2x} + c_2 e^{5x}}_{y_h} + \underbrace{6e^x}_{y_p}$$