Math 2150
9/4/24

Topic 3 – First order linear ODEs

A first order linear ODE is an equation of the form:

$$a_1(x)y' + a_0(x)y = g(x)$$

If we are considering an interval I where we want to find the solution and $a_1(x) \neq 0$ for all x in I, then we can divide through by $a_1(x)$ to get

$$y' + a(x)y = b(x)$$

where $a(x) = \dfrac{a_0(x)}{a_1(x)}$ and $b(x) = \dfrac{g(x)}{a_1(x)}$

This is the kind of equation we will solve.

Suppose we want to solve

$$\boxed{y' + a(x)y = b(x)} \quad (*)$$

on some interval I where $a(x), b(x)$ are continuous on I.

Since $a(x)$ is continuous on I we can create an anti-derivative

<u>Ex:</u>

$$y' + \underbrace{2x}_{a(x)=2x}\, y = \overbrace{x}^{b(x)=x}$$

$$A(x) = \int 2x\,dx = x^2$$

$A(x) = \int a(x)\,dx$ on I.

So, $A'(x) = a(x)$.

Multiply $(*)$ by $e^{A(x)}$ ← (integrating factor)

to get:

$$\underbrace{e^{A(x)} \cdot y' + e^{A(x)} \cdot a(x) y}_{(e^{A(x)} y)'} = e^{A(x)} b(x$$

$$(fg)' = f'g + fg'$$

So we get

$$\left(e^{A(x)} y\right)' = e^{A(x)} b(x)$$

Let $B(x) = \int e^{A(x)} b(x)\,dx$

be an anti-derivative of $e^{A(x)} b(x)$.

So by integrating we get

$$e^{A(x)} y = B(x) + C$$

where C is a constant

Thus,

$$y = e^{-A(x)} \cdot B(x) + Ce^{-A(x)}$$

So y must be of this form.
Since you can reverse the steps above any function of this form will be a solution and this is the general solution.

Ex: Solve

$$y' + 2xy = x$$

on $I = (-\infty, \infty)$.

$$y' + \boxed{2x}\, y = x$$

Let $A(x) = \int 2x\,dx$

$$= x^2.$$

multiply by $e^{A(x)} = e^{x^2}$ to get:

$$e^{x^2} y' + \underbrace{e^{x^2} \cdot 2x}_{\frac{d}{dx} e^{x^2}}\, y = e^{x^2} \cdot x$$

This gives

$$\left(e^{x^2} \cdot y\right)' = x e^{x^2}$$

So,

$$e^{x^2} y = \int x e^{x^2}\, dx$$

Thus,

$$e^{x^2} y = \frac{1}{2} e^{x^2} + C$$

$$\int x e^{x^2}\, dx = \frac{1}{2}\int e^{u}\, du$$

$$u = x^2$$
$$du = 2x\,dx$$
$$\tfrac{1}{2}\, du = x\,dx$$

$$= \frac{1}{2} e^{u} + C$$
$$= \frac{1}{2} e^{x^2} + C$$

Multiply both sides by e^{-x^2} to get

$$\underbrace{e^{-x^2} e^{x^2}}_{1} y = \frac{1}{2} \underbrace{e^{-x^2} e^{x^2}}_{1} + C e^{-x^2}$$

$$e^{-x^2} e^{x^2} = e^{-x^2 + x^2} = e^0 = 1$$

We get

$$\boxed{y = \frac{1}{2} + C e^{-x^2}}$$

This is the general solution to

$$y' + 2xy = x$$

on $I = (-\infty, \infty)$.

Ex: Let's solve

$$y' + \underbrace{\cos(x)}_{a(x) = \cos(x)} y = \sin(x)\cos(x)$$

on $I = (-\infty, \infty)$.

Let $A(x) = \int \cos(x)\,dx = \sin(x)$.

Multiply by $e^{A(x)} = e^{\sin(x)}$ to get:

$$\underbrace{e^{\sin(x)} y' + e^{\sin(x)} \cos(x)\, y}_{\left(e^{A(x)} y\right)'} = e^{\sin(x)} \sin(x)\cos(x)$$

We get

$$\left(e^{\sin(x)} y\right)' = e^{\sin(x)} \sin(x)\cos(x)$$

So,

$$e^{\sin(x)} y = \int e^{\sin(x)} \sin(x)\cos(x)\,dx$$

$$\int e^{\sin(x)} \sin(x)\cos(x)\,dx =$$

$$= \int e^t \cdot t\,dt = \int t e^t\,dt$$

$t = \sin(x)$

$dt = \cos(x)\,dx$

$$= te^t - \int e^t\,dt$$

$u = t$ $\quad du = dt$

$dv = e^t\,dt$ $\quad v = e^t$

$\int u\,dv = uv - \int v\,du$

$$= te^t - e^t + C$$

$$= \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

So,

$$e^{\sin(x)} y = \sin(x) e^{\sin(x)} - e^{\sin(x)} + C$$

Thus,

$$\boxed{y = \sin(x) - 1 + Ce^{-\sin(x)}}$$

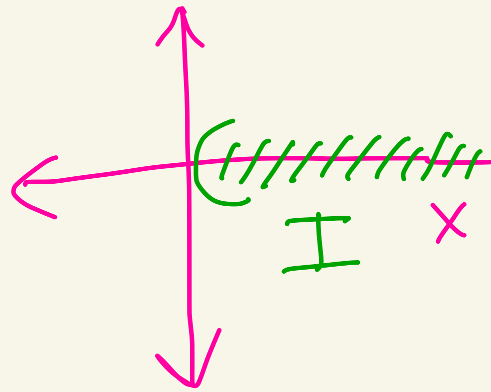
$$\boxed{\frac{1}{e^{\sin(x)}} = e^{-\sin(x)}}$$

is the general solution.

Ex: Solve

$$xy' + y = 3x^3 + 1$$

on $I = (0, \infty)$.



Divide by x to get

$$y' + \frac{1}{x}y = 3x^2 + \frac{1}{x}$$

aren't defined at $x = 0$. So thats why we will solve for $x > 0$ or $I = (0, \infty)$

Let $A(x) = \int \frac{1}{x}\,dx$

$= \ln(x)$.

We multiply by

$$e^{A(x)} = e^{\ln(x)} = x$$

to get

$$x\left(y' + \frac{1}{x}y\right) = x\left(3x^2 + \frac{1}{x}\right)$$

that is

$$xy' + y = 3x^3 + 1$$

This is

$$\underbrace{(xy)}_{\left(e^{A(x)}y\right)'}{}' = 3x^3 + 1$$

Integrate both sides to get

$$xy = \int (3x^3 + 1)\,dx$$

So,

$$xy = \frac{3}{4}x^4 + x + C$$

Thus,

$$\boxed{y = \frac{3}{4}x^3 + 1 + \frac{C}{x}}$$

is the general solution to $xy' + y = 3x^3 + 1$ on $I = (0, \infty)$.

<u>Ex:</u> Solve the initial-value problem

$$\boxed{\begin{aligned} &xy' + y = 3x^3 + 1 \\ &y(1) = 2 \end{aligned}}$$

on $I = (0, \infty)$

We saw that the general sol to $xy' + y = 3x^3 + 1$ is

$$y = \frac{3}{4}x^3 + 1 + \frac{c}{x}$$

We want $y(1) = 2$.

Plug in $x=1$, $y=2$ into to get

$$2 = y(1) = \frac{3}{4}(1)^3 + 1 + \frac{c}{1}$$

So,

$$2 = \frac{7}{4} + C$$

So,

$$\frac{1}{4} = C$$

Thus, the solution is

$$y = \frac{3}{4}x^3 + 1 + \frac{(1/4)}{x}$$

or

$$\boxed{y = \frac{3}{4}x^3 + 1 + \frac{1}{4x}}$$