Math 2150
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Topic 4 – Separable first order ODEs

<u>Def:</u> A first order ODE is called <u>separable</u> if it is of the form

$$\underbrace{N(y)}_{\text{function with no } x\text{'s}} \cdot y' = \underbrace{M(x)}_{\text{function with no ys}}$$

or

$$N(y) \cdot \frac{dy}{dx} = M(x)$$

Ex: $\underbrace{y^2}_{N(y)} \frac{dy}{dx} = \underbrace{x-5}_{M(x)}$ ← separable first order non-linear

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Ex: $y' = \frac{x^2}{y}$

Multiply by y to get

$$\underbrace{y}_{M(y)} \cdot y' = \underbrace{x^2}_{N(x)}$$ ← separable first order non-linear

How to solve a separable ODE

Formal way

$$N(y)\cdot y' = M(x)$$

$$\downarrow$$

$$N(y(x))\cdot y'(x) = M(x)$$

$$\downarrow$$

$$\int N(y(x))\cdot y'(x)\,dx = \int M(x)\,dx$$

$$\downarrow \quad \begin{aligned} u &= y(x) \\ du &= y'(x)\,dx \end{aligned}$$

$$\int N(u)\,du = \int M(x)\,dx$$

Now integrate.
Remember $u = y$.

Informal way

$$N(y)\cdot \frac{dy}{dx} = M(x)$$

$$\downarrow$$

$$N(y)\,dy = M(x)\,dx$$

[informal differential form notation]

$$\downarrow$$

$$\int N(y)\,dy = \int M(x)\,dx$$

Now integrate

Ex: Find a solution to

$$y^2 \frac{dy}{dx} = x - 5$$

Also, on what interval does our solution exist?

We have:

$$y^2 \frac{dy}{dx} = x - 5$$

$$y^2\, dy = (x-5)\, dx$$

$$\int y^2\, dy = \int (x-5)\, dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$$

$$y^3 = \frac{3}{2}x^2 - 15x + D \quad \text{where } D = 3C$$

$$y = \left(\frac{3}{2}x^2 - 15x + D\right)^{1/3}$$

Thus, a solution to $y^2 \frac{dy}{dx} = x - 5$

is given by

$$y = \left(\frac{3}{2}x^2 - 15x + D\right)^{1/3}$$

where D is any constant.

This solution works on

$$I = (-\infty, \infty)$$

all x's are ok to plug into the solution

Ex: Find a solution to

$$\frac{dy}{dx} + 2xy = 0$$

linear eqn. we solved it last time with a different method

On what interval I does the solution exist?

We have

$$\frac{dy}{dx} + 2xy = 0$$

$$\frac{dy}{dx} = -2xy$$

$$\frac{1}{y}\,dy = -2x\,dx$$

$$\int \frac{1}{y}\,dy = -\int 2x\,dx$$

$$\ln|y| = -x^2 + C_1$$

$$e^{\ln|y|} = e^{-x^2+C_1}$$

$e^{\ln(t)} = t$

$e^{A+B} = e^A e^B$

$$|y| = e^{-x^2} \cdot e^{C_1}$$

$$|y| = C_2 e^{-x^2} \quad \text{where } C_2 = e^{C_1}$$

$$y = \pm C_2 e^{-x^2}$$

$$y = Ce^{-x^2} \quad \text{where } C \text{ is a constant}$$

$$(C = \pm C_2)$$

So, $y = Ce^{-x^2}$ which exists on $I = (-\infty, \infty)$ is a solution to

any x is ok to plug into $y = Ce^{-x^2}$

$$\frac{dy}{dx} + 2xy = 0.$$

This is the same answer as last time in Topic 3.

Topic 5 – First order exact equations

Suppose you have a first-order equation of the form:

$$\underbrace{M(x,y)}+\underbrace{N(x,y)}\cdot y'=0$$

expressions with numbers, xs, and ys.

Further suppose there exists a function $f(x,y)$ where

$$\frac{\partial f}{\partial x}=M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y}=N(x,y)$$

Then,

$$M(x,y)+N(x,y)\cdot y'=0$$

becomes

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$

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$f(x,y)$ is a

This is equivalent to

$$\frac{df}{dx} = 0$$

function of x,y

$y = y(x)$ is a function of x

chain rule:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{d}{dx}(x) + \frac{\partial f}{\partial y} \cdot \frac{d}{dx}(y)$$

$$= \frac{\partial f}{\partial x} \cdot (1) + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

So, for example the family of curves given by

$$f(x,y) = c$$

where c is a constant will satisfy $\frac{df}{dx} = 0$.

<u>Summary:</u> If $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$, then the family of curves $f(x,y) = c$ where c is any constant will give implicit solutions to $M(x,y) + N(x,y) \cdot y' = 0$

When such an f exists we call the equation $M(x,y) + N(x,y) \cdot y' = 0$ an <u>exact equation</u>.