Math 2550-01

8/26/24

Topic 1 – Vectors

Def: Let $n \geq 1$ be an integer.

[So, $n = 1, 2, 3, 4, 5, 6, \ldots$]

An n-dimensional real vector is a list of n numbers.

We use brackets < and > to denote vectors and commas to separate the numbers in the list.

We use an arrow over a variable to denote a vector, such as $\vec{v}$.

Ex: Some 2-dimensional vectors:

$\langle 2, 4 \rangle$

$\langle 4, 2 \rangle$

different vectors
order matters

$\langle -\frac{1}{3}, \pi \rangle$

Ex: Some 3-dim. vectors:

$\langle 0, 0, 0 \rangle$

$\langle 2, -1, 5 \rangle$

Ex: Some 6-dim. vectors:

$\langle -1, 0, 3, 4, -\frac{1}{2}, 2.7 \rangle$

$\langle 0, 1, 0, 1, 0, 1 \rangle$

Def: Let $n \geq 1$ be an integer. Define $\mathbb{R}^n$ to be the set of all n-dimensional real vectors.
That is,

$$\mathbb{R}^n = \underbrace{\{\langle a_1, a_2, \ldots, a_n \rangle \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\}}$$

read: the set of all vectors $\langle a_1, a_2, \ldots, a_n \rangle$ where $a_1, a_2, \ldots, a_n$ are real numbers.

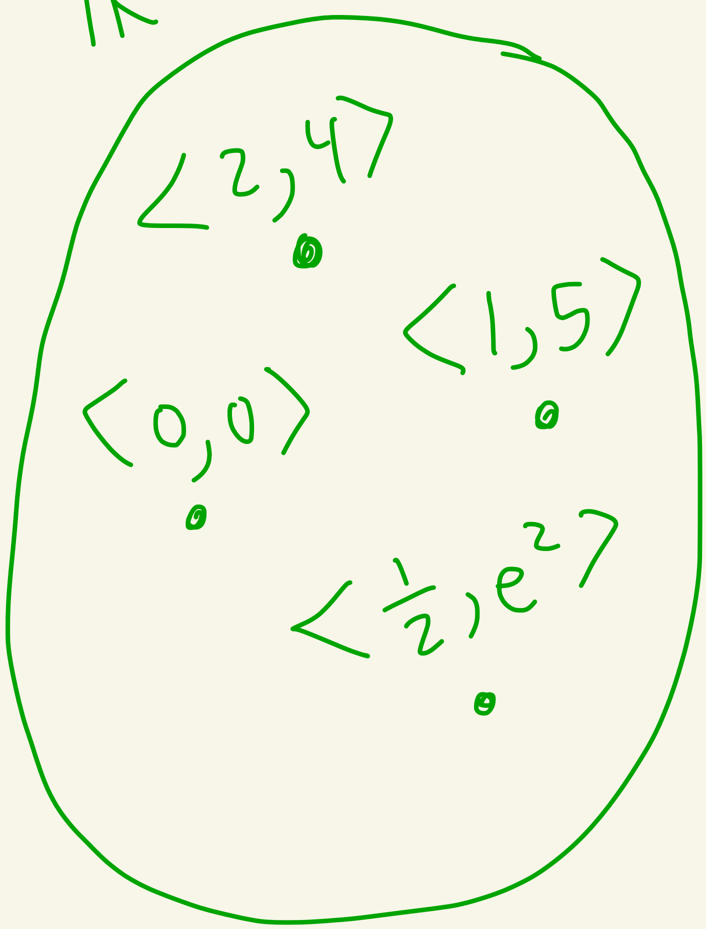
Ex: $(n=2)$

$$\mathbb{R}^2 = \{ \langle a_1, a_2 \rangle \mid a_1, a_2 \in \mathbb{R} \}$$

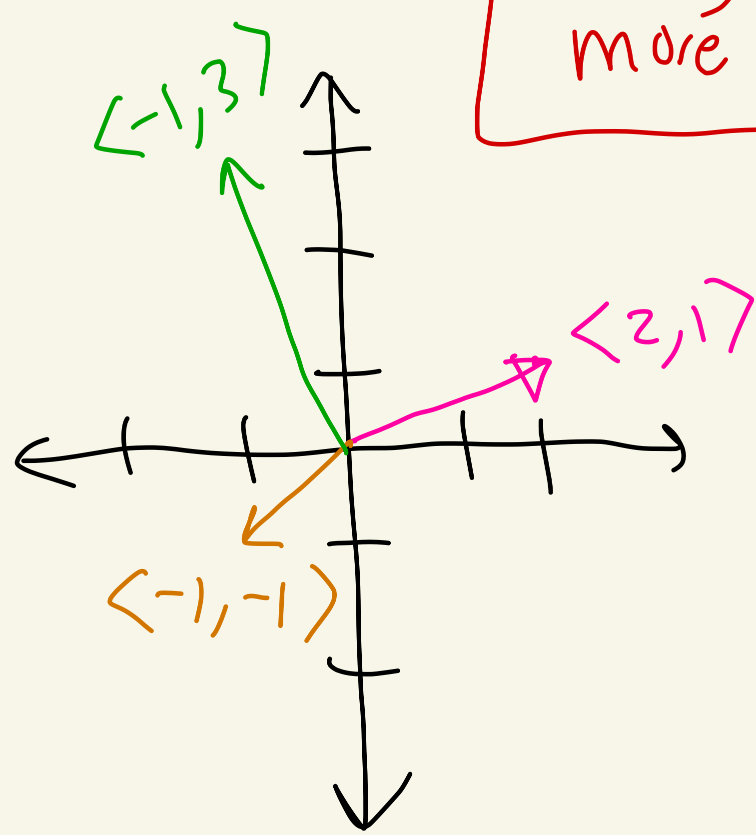
$$= \{ \underbrace{\langle 2,4 \rangle}_{\substack{a_1 = 2 \\ a_2 = 4}}, \underbrace{\langle 0,0 \rangle}_{\substack{a_1 = 0 \\ a_2 = 0}}, \underbrace{\langle \tfrac{1}{2}, e^2 \rangle}_{\substack{a_1 = 1/2 \\ a_2 = e^2}}, \ldots \}$$

infinitely many more

$\mathbb{R}^2$



$\mathbb{R}^2$

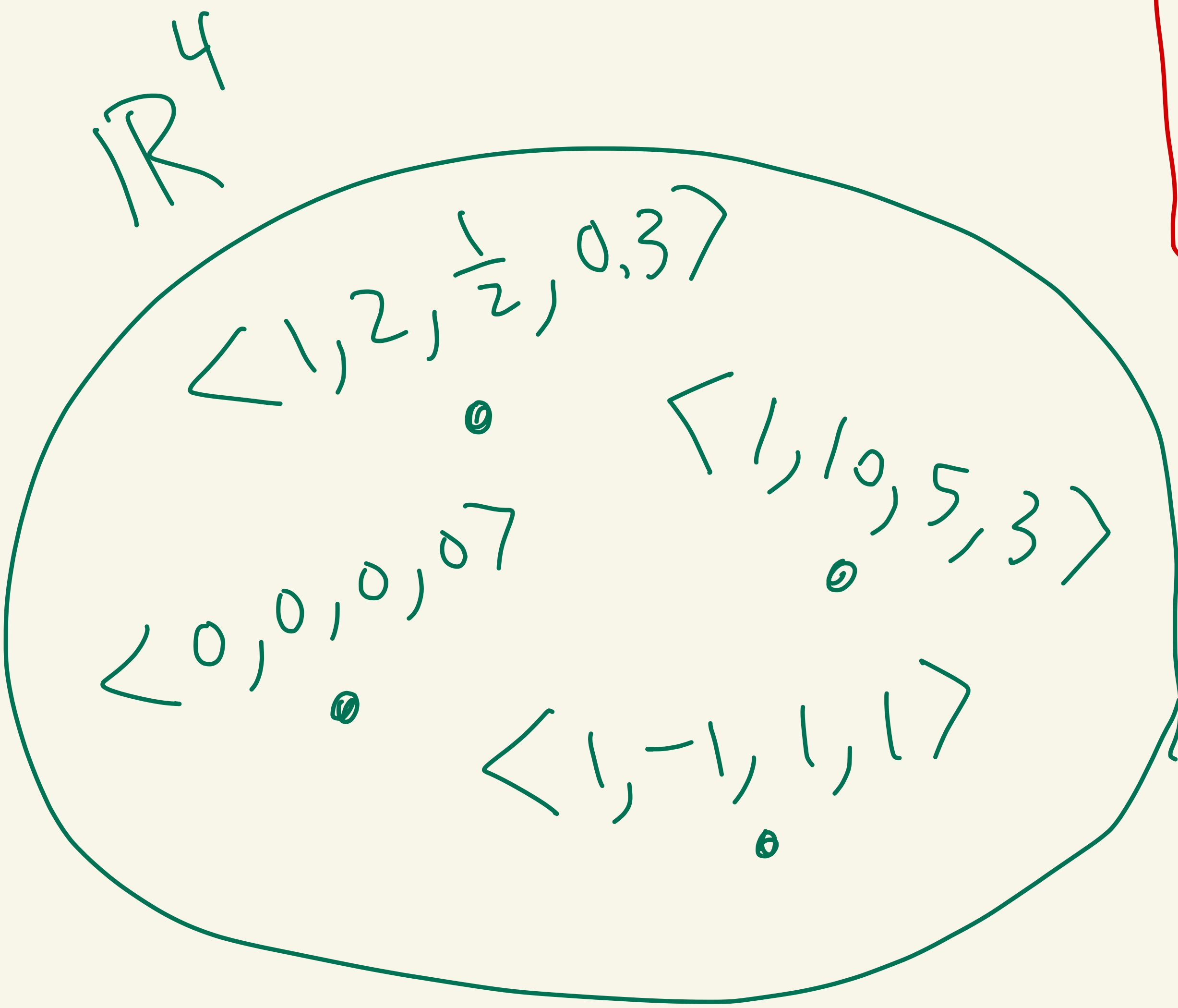


Ex:

$$\mathbb{R}^4 = \{\langle a_1, a_2, a_3, a_4\rangle \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}$$

$$= \{\langle 1, 2, \tfrac{1}{2}, 0.3\rangle, \langle 1, 1, 1, 1\rangle, \langle 5, \pi^3, 0.27136, 13\rangle, \dots\}$$

infinitely many more



Def: Let $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ be a vector in $\mathbb{R}^n$.
The length (or norm or magnitude) of $\vec{v}$ is

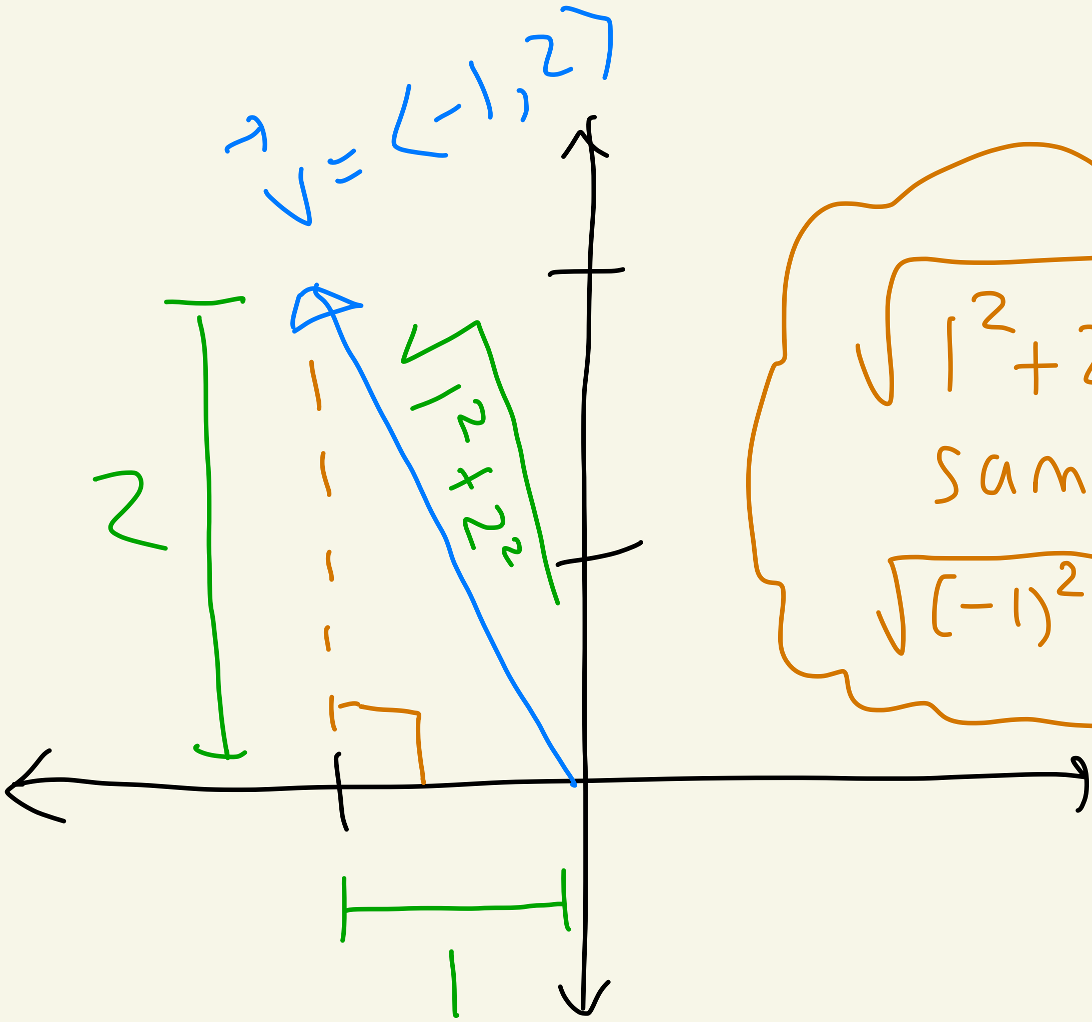
$$\underbrace{\|\vec{v}\|} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

some people use $|\vec{v}|$ instead of $\|\vec{v}\|$

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle -1, 2 \rangle$

Then, $\|\vec{v}\| = \sqrt{(-1)^2 + (2)^2} = \sqrt{5}$

$\approx 2.236\ldots$



$\sqrt{1^2 + 2^2}$ same as $\sqrt{(-1)^2 + 2^2}$

Ex: In $\mathbb{R}^6$ let $\vec{v} = \langle 0, 1, -2, -3, 10, 5 \rangle$

Then,

$$\|\vec{v}\| = \sqrt{0^2 + 1^2 + (-2)^2 + (-3)^2 + 10^2 + 5^2}$$

$$= \sqrt{1 + 4 + 9 + 100 + 25} = \sqrt{139}$$

$\approx 11.7898\ldots$

Operations on vectors

Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$

Let α be a $\underbrace{\text{scalar}}_{\text{number}}$ in $\mathbb{R}$

Some greek letters

α - alpha

β - beta

ϕ - phi

θ - theta

γ - gamma

δ - delta

Suppose

$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ and

$\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle$.

Then define:

$$\vec{v}+\vec{w} = \langle a_1+b_1, a_2+b_2, \ldots, a_n+b_n \rangle$$

$$\vec{v}-\vec{w} = \langle a_1-b_1, a_2-b_2, \ldots, a_n-b_n \rangle$$

$$\underbrace{\alpha\vec{v}}_{\text{scaling a vector}} = \langle \alpha a_1, \alpha a_2, \ldots, \alpha a_n \rangle$$

Ex: In $\mathbb{R}^2$ we have:

$$\langle 5,-2\rangle + \langle 1,3\rangle = \langle 5+1, -2+3\rangle$$
$$= \langle 6,1\rangle$$

$$3\langle 5,-2\rangle = \langle 3(5), 3(-2)\rangle$$
$$= \langle 15,-6\rangle$$

$$\langle 5,-2\rangle - \langle 1,3\rangle = \langle 5-1, -2-3\rangle$$
$$= \langle 4,-5\rangle$$

Ex: In $\mathbb{R}^5$ we have:

$$\langle 1,0,3,4,-1\rangle - 2\langle 0,2,1,4,8\rangle$$

$$= \langle 1,0,3,4,-1\rangle - \langle 2(0), 2(2), 2(1), 2(4), 2(8)\rangle$$
$$= \langle 1,0,3,4,-1\rangle - \langle 0,4,2,8,16\rangle$$
$$= \langle 1-0, 0-4, 3-2, 4-8, -1-16\rangle$$

$= \langle 1, -4, 1, -4, -17 \rangle$

Notation: In $\mathbb{R}^n$, the zero vector, denoted by $\vec{0}$, is the vector containing all 0's.

In $\mathbb{R}^2$, $\vec{0} = \langle 0, 0 \rangle$.

In $\mathbb{R}^3$, $\vec{0} = \langle 0, 0, 0 \rangle$

In $\mathbb{R}^4$, $\vec{0} = \langle 0, 0, 0, 0 \rangle$

And so on...

Properties of vectors.

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$

Let α, β be $\underbrace{\text{scalars}}_{\text{numbers}}$ in $\mathbb{R}$.

Then:

① $\vec{u} + \vec{w} = \vec{w} + \vec{u}$ (commutative)

② $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (associative

③ $\alpha(\beta \vec{u}) = (\alpha\beta)\vec{u}$

Ex: $2(5\vec{u}) = (10)\vec{u}$ — $\alpha = 2$, $\beta = 5$

④ $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$

$5\vec{u} = 2\vec{u} + 3\vec{u}$

⑤ $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$

$2(\vec{u} + \vec{v}) = 2\vec{u} + 2\vec{v}$

⑥ $\vec{u} + \vec{0} = \vec{u}$

$\vec{0} + \vec{u} = \vec{u}$

⑦ $\vec{u} + (-\vec{u}) = \vec{0}$

$(-\vec{u}) + \vec{u} = \vec{0}$

proof that 4 is true when $n=2$

Let $\vec{u}$ be in $\mathbb{R}^2$.
Let α, β be scalars/numbers.
Then, $\vec{u} = \langle a_1, a_2 \rangle$ where a_1, a_2 are scalars/numbers.

Then,

$$
\begin{aligned}
(\alpha+\beta)\vec{u} &= (\alpha+\beta)\langle a_1, a_2 \rangle \\
&= \langle (\alpha+\beta)a_1, (\alpha+\beta)a_2 \rangle \\
&= \langle \alpha a_1 + \beta a_1, \alpha a_2 + \beta a_2 \rangle \\
&= \langle \alpha a_1, \alpha a_2 \rangle + \langle \beta a_1, \beta a_2 \rangle \\
&= \alpha \langle a_1, a_2 \rangle + \beta \langle a_1, a_2 \rangle \\
&= \alpha \vec{u} + \beta \vec{u}. \quad \square
\end{aligned}
$$