Math 2550-04
10/16/24

Topic 6 - Coordinate systems in $\mathbb{R}^n$

beta

<u>Def:</u> Let $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be r vectors in $\mathbb{R}^n$.

- We say that a vector $\vec{v}$ is in the <u>span</u> of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ if we can write

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_r \vec{v}_r$$

where $c_1, c_2, \ldots, c_r$ are real numbers.

- The expression

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_r \vec{v}_r$$

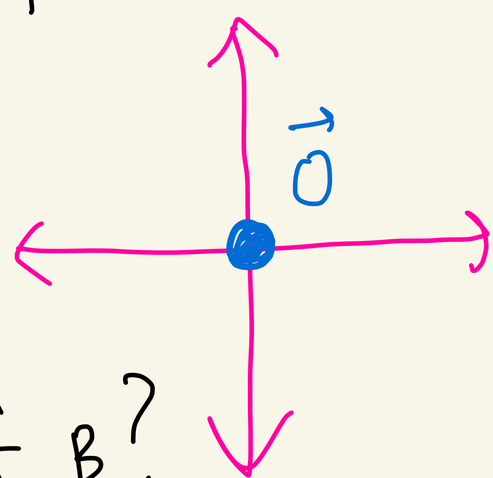
is called a <u>linear combination</u> of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$

- If $r=1$, that is, if $\beta=\{\vec{v}_1\}$, then we say that β is a <u>linearly dependent</u> set if $\vec{v}_1=\vec{0}$ [or we just say $\vec{v}_1$ is linearly dependent]
  If $\vec{v}_1 \neq \vec{0}$, then β is called <u>linearly independent</u>.

- If $r \geq 2$, that is if β has 2 or more vectors, then we say that β is a <u>linearly dependent</u> set if

One of the vectors in β can be written as a linear combination of the other vectors. If this isn't the case then we say that β is <u>linearly independent</u>

<u>Ex:</u> In $\mathbb{R}^2$, let $\beta = \{\vec{0}\}$.



β is a linearly dependent set by def.

What's in the span of β?

Any vector that looks like

$$c_1 \cdot \vec{0}$$

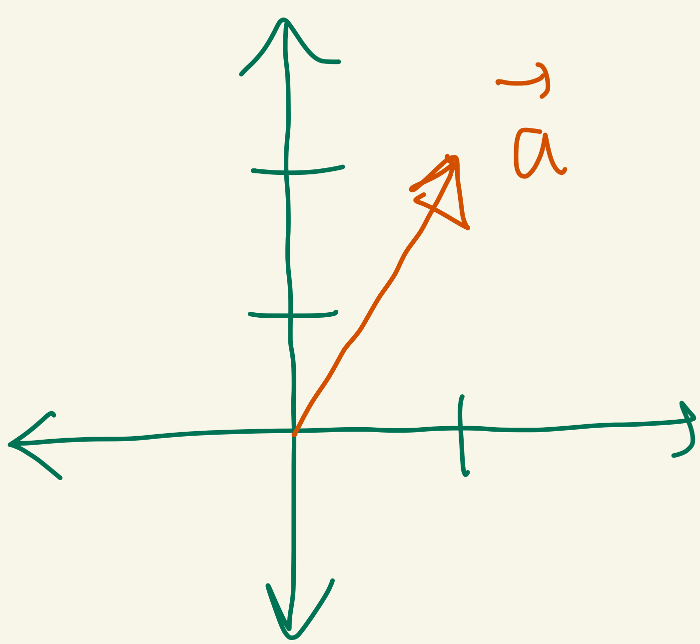
where c_1 is a number.

We always get $c_1 \cdot \vec{0} = \vec{0}$.

So the only vectors in the span of β are $\vec{0}$.

<u>Ex:</u> Consider $\vec{a} = \langle 1, 2 \rangle$ in $\mathbb{R}^2$. Let $\beta = \{\vec{a}\}$.

Since $\vec{a} \neq \vec{0}$ we say that β is a linearly independent set.



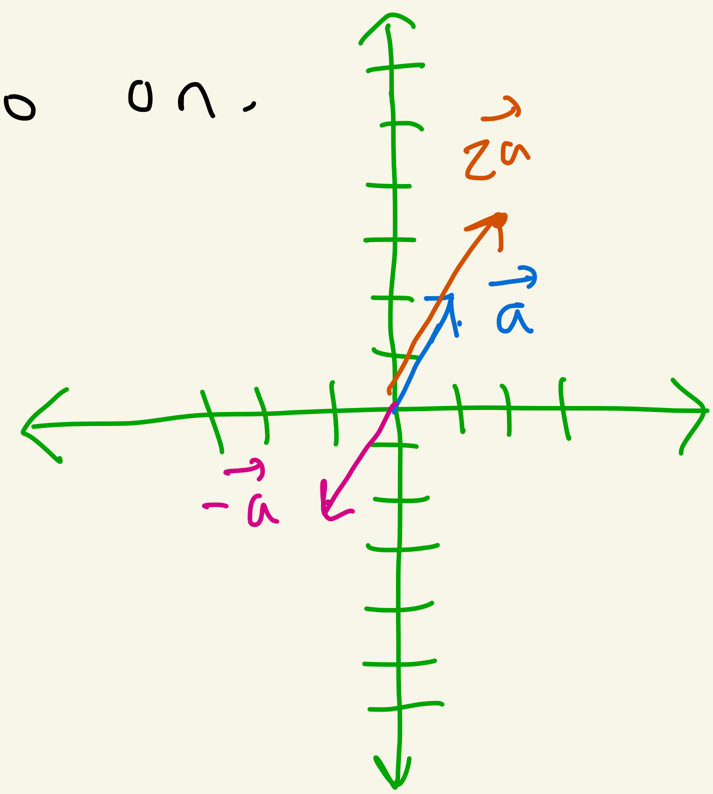
What is in the span of $\vec{a}$?

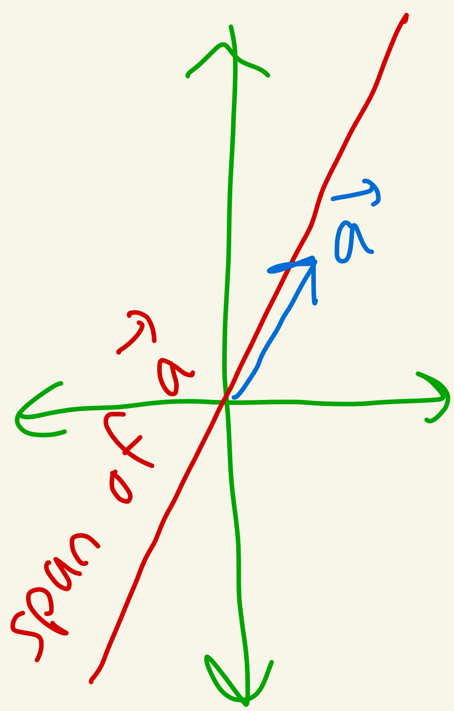
All the multiples $c_1\vec{a}$ of $\vec{a}$.

For example,

$$\left.\begin{aligned} 1\cdot\vec{a} &= \langle 1, 2\rangle \\ -1\cdot\vec{a} &= \langle -1, -2\rangle \\ 2\cdot\vec{a} &= \langle 2, 4\rangle \\ -3\cdot\vec{a} &= \langle -3, -6\rangle \end{aligned}\right\}\text{ these are all in the span of } \vec{a}$$

and so on.





Ex: Consider $\mathbb{R}^2$. Let $\vec{v}_1 = \langle 1, 1 \rangle$ and $\vec{v}_2 = \langle 2, 2 \rangle$.

Let $\beta = \{ \vec{v}_1, \vec{v}_2 \}$.

Q: What are some vectors in the span of β?

Anything that looks like

$$c_1 \vec{v}_1 + c_2 \vec{v}_2$$

For example,

$$2 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 = 2\langle 1,1 \rangle + \langle 2,2 \rangle = \boxed{\langle 4,4 \rangle}$$

$$-1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 = -\langle 1,1 \rangle + 0\langle 2,2 \rangle = \boxed{\langle -1,-1 \rangle}$$

So, $\langle 4,4 \rangle$ and $\langle -1,-1 \rangle$ are in the span of $\beta = \{ \vec{v}_1, \vec{v}_2 \}$.

Note that in general a vector in the span of $\beta = \{\vec{V}_1, \vec{V}_2\}$ is of the form

$$\begin{aligned} c_1\vec{V}_1 + c_2\vec{V}_2 &= c_1\langle 1,1\rangle + c_2\langle 2,2\rangle \\ &= c_1\langle 1,1\rangle + 2c_2\langle 1,1\rangle \\ &= \underbrace{(c_1+2c_2)}_{\text{a number}}\langle 1,1\rangle \\ &= (c_1+2c_2)\cdot\vec{V}_1 \end{aligned}$$

So any vector in the span of $\vec{V}_1, \vec{V}_2$ is actually just in the span of $\vec{V}_1$ only.

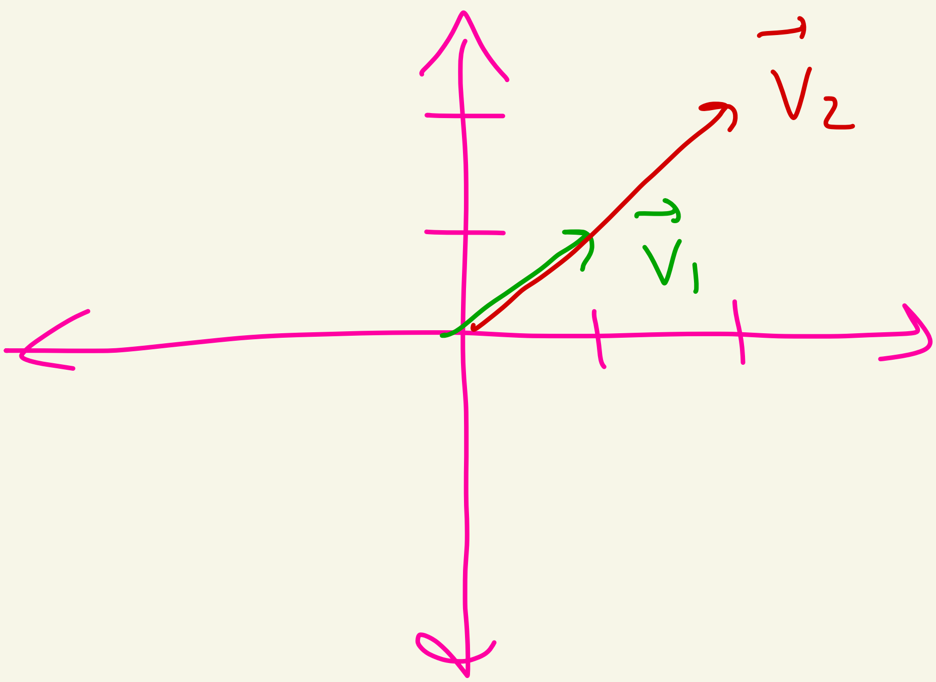
$$\boxed{\begin{aligned} \vec{V}_1 &= \langle 1,1\rangle \\ \vec{V}_2 &= \langle 2,2\rangle \end{aligned}}$$

Note that

$$\vec{V}_2 = 2\vec{V}_1$$

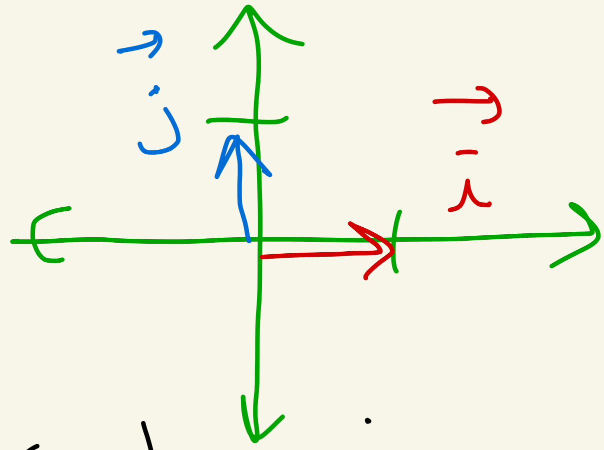
Since $\vec{v}_2$ is a linear combination of $\vec{v}_1$ we say that $\beta = \{\vec{v}_1, \vec{v}_2\}$ is linearly dependent.

Note you can write $\vec{v}_2 = 2\vec{v}_1$ as $2\vec{v}_1 - 1 \cdot \vec{v}_2 = \vec{0}$



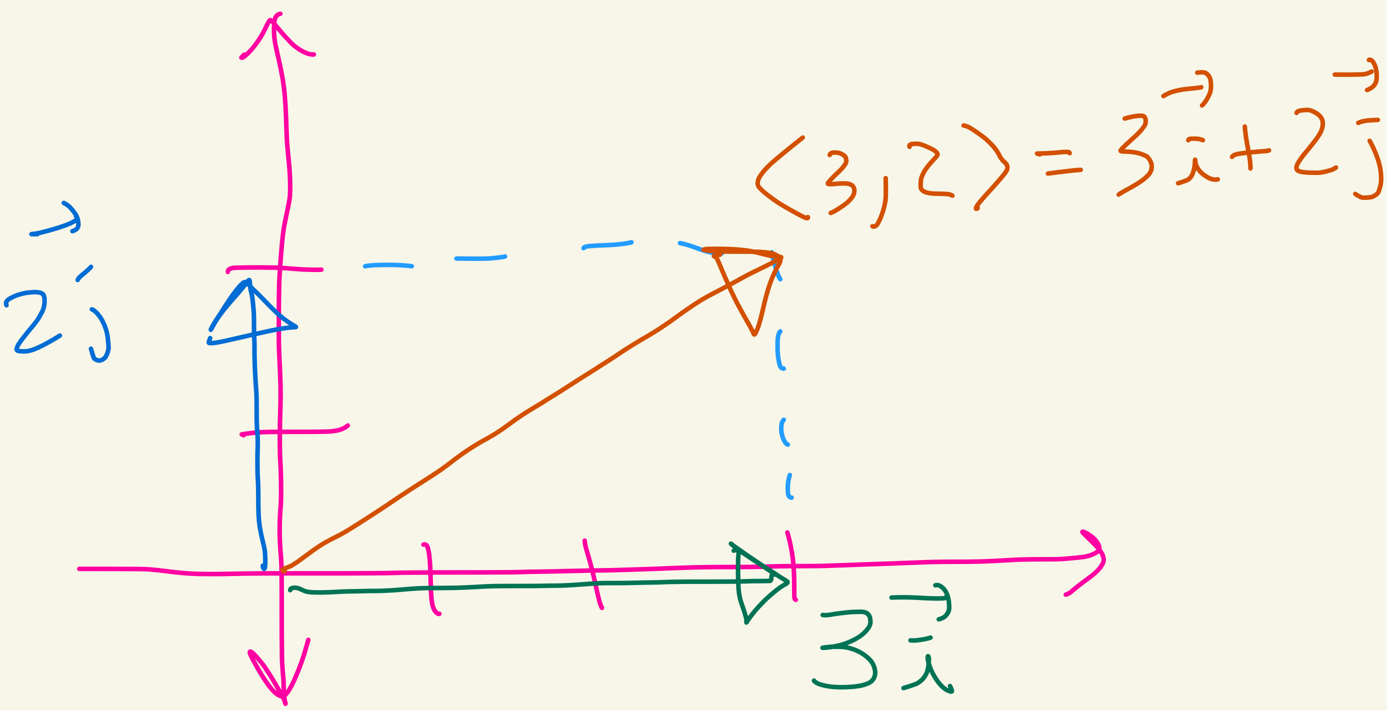
Ex: Let $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$ in $\mathbb{R}^2$.

Let $\beta = \{\vec{i}, \vec{j}\}$



What are some vectors in the span of β?

$$3\vec{i} + 2\vec{j} = 3\langle 1, 0 \rangle + 2\langle 0, 1 \rangle = \boxed{\langle 3, 2 \rangle}$$



$$-\vec{i}+2\vec{j} = -\langle 1,0\rangle + 2\langle 0,1\rangle$$
$$= \boxed{\langle -1,2\rangle}$$

So, $\langle 3,2\rangle$, $\langle -1,2\rangle$ are in the span of β.

In general, any vector $\langle a,b\rangle$ is in the span of β since

$$\langle a,b\rangle = \langle a,0\rangle + \langle 0,b\rangle$$
$$= a\langle 1,0\rangle + b\langle 0,1\rangle$$
$$= a\vec{i} + b\vec{j}$$

Q: Is $\beta = \{\vec{i},\vec{j}\}$ a linearly dependent or independent set?

Can we make $\vec{i}$ from $\vec{j}$?

That is, is $\vec{i}$ a lin. combo. of $\vec{j}$?

We are asking can we write

$$\vec{i} = c_1 \vec{j} \; ?$$

If so, then

$$\underbrace{\langle 1, 0 \rangle}_{\vec{i}} = c_1 \underbrace{\langle 0, 1 \rangle}_{\vec{j}}$$

This would require

$$\langle 1, 0 \rangle = \langle 0, c_1 \rangle$$

But then $1 = 0$ which can't happen.

Can we make $\vec{j}$ from $\vec{i}$ as a lin. combo. ?

We would need

$$\vec{j} = c_1 \vec{i}$$

This would require

$$\underbrace{\langle 0, 1 \rangle}_{\vec{j}} = c_1 \underbrace{\langle 1, 0 \rangle}_{\vec{i}}$$

This would require

$$\langle 0, 1 \rangle = \langle c_1, 0 \rangle$$

But then $1 = 0$ which can't happen.

So, $\beta = \{ \vec{i}, \vec{j} \}$ are linearly independent.

Syllabus –

test 1 – 33.3%
test 2 – 33.3%
final – 33.3%

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drop 1 –

$\max\{\text{test 1}, \text{test 2}\} = 50\%$

final = 50%

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no final –

test 1 – 50%
test 2 – 50%

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final

final – 100%