Math 2550-04
10/2/24

M	W
	10/2 Finish topic 5
10/7 Review	10/9 Test 1

Ex: Last time we showed that

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} = -1$$

by expanding on column 3. Let's compute this again but expand on row 2.

$$\det\begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix} \qquad \begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix}$$

$$= (-1)(-2)\begin{vmatrix}1 & 0\\ 4 & -2\end{vmatrix} + (1)(-4)\begin{vmatrix}3 & 0\\ 5 & -2\end{vmatrix} + (-1)(3)\begin{vmatrix}3 & 1\\ 5 & 4\end{vmatrix}$$

$$\begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix} \qquad \begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix} \qquad \begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix}$$

$$\begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix} \qquad \begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix} \qquad \begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix}$$

$$= 2\left[(1)(-2) - (0)(4)\right]$$

$$-4\left[(3)(-2) - (0)(5)\right]$$

$$-3\left[(3)(4)-(1)(5)\right]$$

$$= 2[-2] - 4[-6] - 3[7]$$

$$= -4 + 24 - 21$$

$$= -1$$

Ex:

$$\det\begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

expand on column 2

$$= (-1)(0)\begin{vmatrix} 2 & 3 & 0 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{vmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$+(1)(1)\begin{vmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{vmatrix} \longleftarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$+(-1)(0)\begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \end{vmatrix} \longleftarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$+(1)(1)\begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & -2 \end{vmatrix} \longleftarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & -2 \end{vmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= \overbrace{(-1)(0)\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix}}^{0} + \overbrace{(1)(0)\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}}^{0} + (-1)(2)\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}.$$

$$+ 0 + \underbrace{(1)(3)\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix}}_{} + 0$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

$$= 0 + 0 - 2\left[(1)(-2) - (-1)(1)\right]$$

$$+ 0 + 3\left[(1)(-2) - (-1)(1)\right] + 0$$

$$= -2\left[-1\right] + 3\left[-1\right] = -1$$

Properties of determinants

Let A and B be $n \times n$ matrices.

① $\det(A^T) = \det(A)$

② $\det(AB) = \det(A) \cdot \det(B)$

③ If $\det(A) = 0$, then A^{-1} does not exist.

④ If $\det(A) \neq 0$, then A^{-1} exists

⑤ If A^{-1} exists, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

$$\boxed{\text{Formula for } A^{-1} \text{ when } A \text{ is } 2\times 2}$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then, $\det(A) = ad - bc$

If $ad - bc \neq 0$, then A^{-1} exists and

$$A^{-1} = \underbrace{\frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{\frac{1}{\det(A)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Ex: $A = \begin{pmatrix} 2 & 6 \\ 7 & -3 \end{pmatrix}$

$\det(A) = (2)(-3) - (6)(7) = -48$

Since $\det(A) \neq 0$, A^{-1} exists

and

$$A^{-1} = \frac{1}{-48}\begin{pmatrix} -3 & -6 \\ -7 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/48 & 6/48 \\ 7/48 & -2/48 \end{pmatrix} = \begin{pmatrix} 1/16 & 1/8 \\ 7/48 & -1/24 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$

$\det(A) = (1)(-6) - (2)(-3) = 0$

So, A^{-1} does not exist

Note: In general, if A is $n \times n$ and $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \cdot M$$

where M is called the "adjugate matrix"