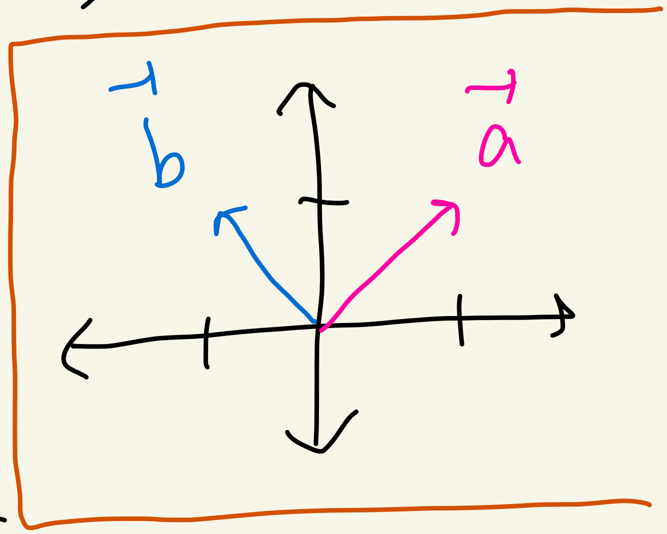
Math 2550-04
10/23/24

Last time we showed that if $\vec{a} = \langle 1,1 \rangle$, $\vec{b} = \langle -1,1 \rangle$ then these vectors are linearly independent



We have 2 linearly independent vectors in $\mathbb{R}^2$. So, they will make a coordinate system / basis.

So every other vector $\vec{v}$ can be written uniquely as

$$\vec{v} = c_1 \vec{a} + c_2 \vec{b}$$

coordinates of $\vec{v}$ with respect to this coordinate system

Let $\beta = [\vec{a}, \vec{b}]$ ← name for coordinate system using $\vec{a}$ & $\vec{b}$

Let $\vec{v} = \langle 3, 1 \rangle$. Let's find $\vec{v}$'s coordinates.

Want to solve

$$\underbrace{\langle 3, 1 \rangle}_{\vec{v}} = \underbrace{c_1 \langle 1, 1 \rangle + c_2 \langle -1, 1 \rangle}_{c_1 \vec{a} + c_2 \vec{b}}$$

This becomes

$$\langle 3, 1 \rangle = \langle c_1, c_1 \rangle + \langle -c_2, c_2 \rangle$$

which gives

$$\langle 3, 1 \rangle = \langle c_1 - c_2, c_1 + c_2 \rangle$$

We get

$$\boxed{\begin{aligned} c_1 - c_2 &= 3 \quad (1) \\ c_1 + c_2 &= 1 \quad (2) \end{aligned}}$$

(1) + (2) gives $2c_1 = 4$. So, $\boxed{c_1 = 2}$

plug $c_1 = 2$ into (2) to get

$2 + c_2 = 1$. So, $\boxed{c_2 = -1.}$

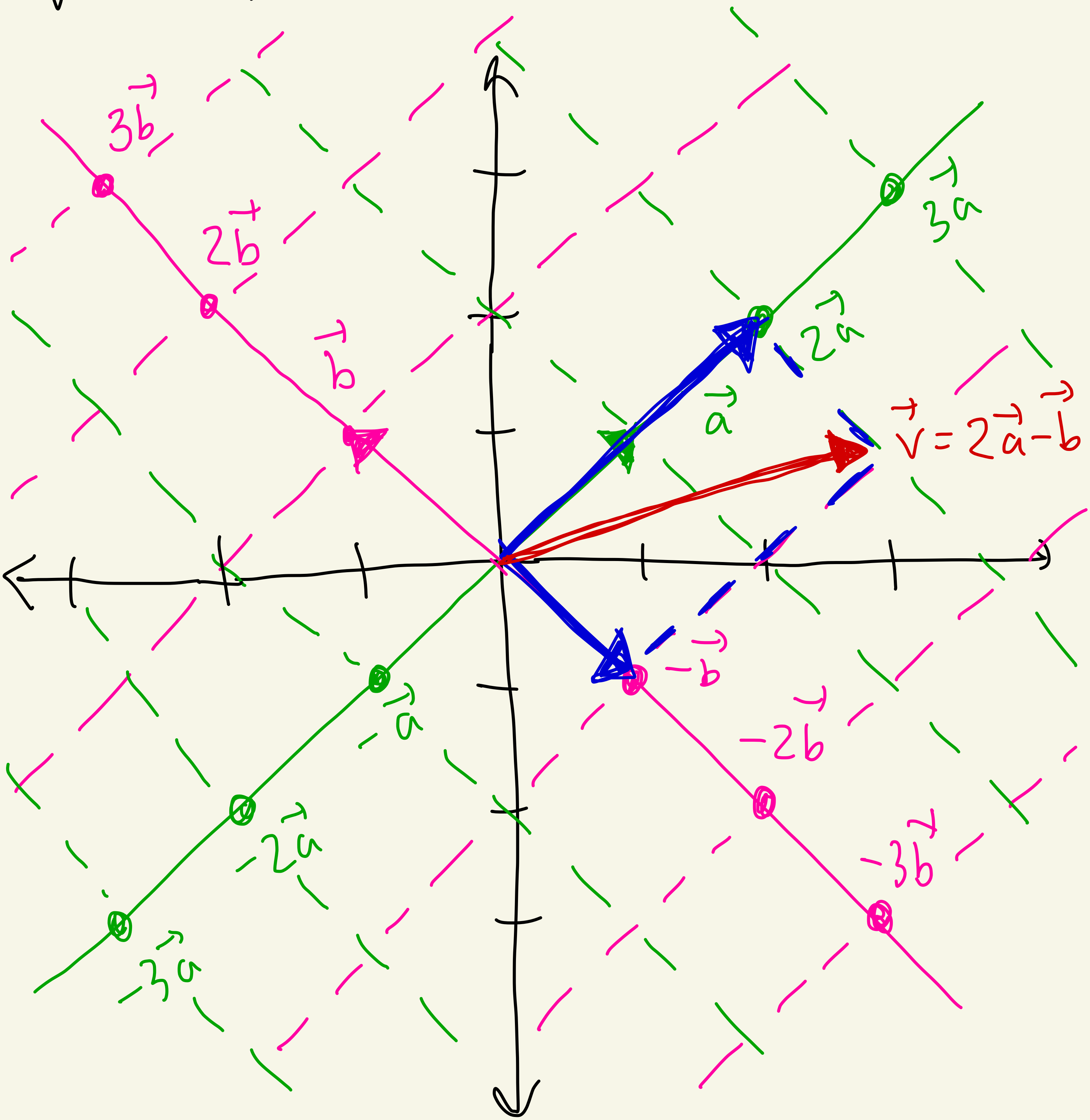
Thus,

$$\underbrace{\langle 3, 1\rangle = 2\langle 1, 1\rangle - 1 \cdot \langle -1, 1\rangle}_{\vec{v} = 2 \cdot \vec{a} - 1 \cdot \vec{b}}$$

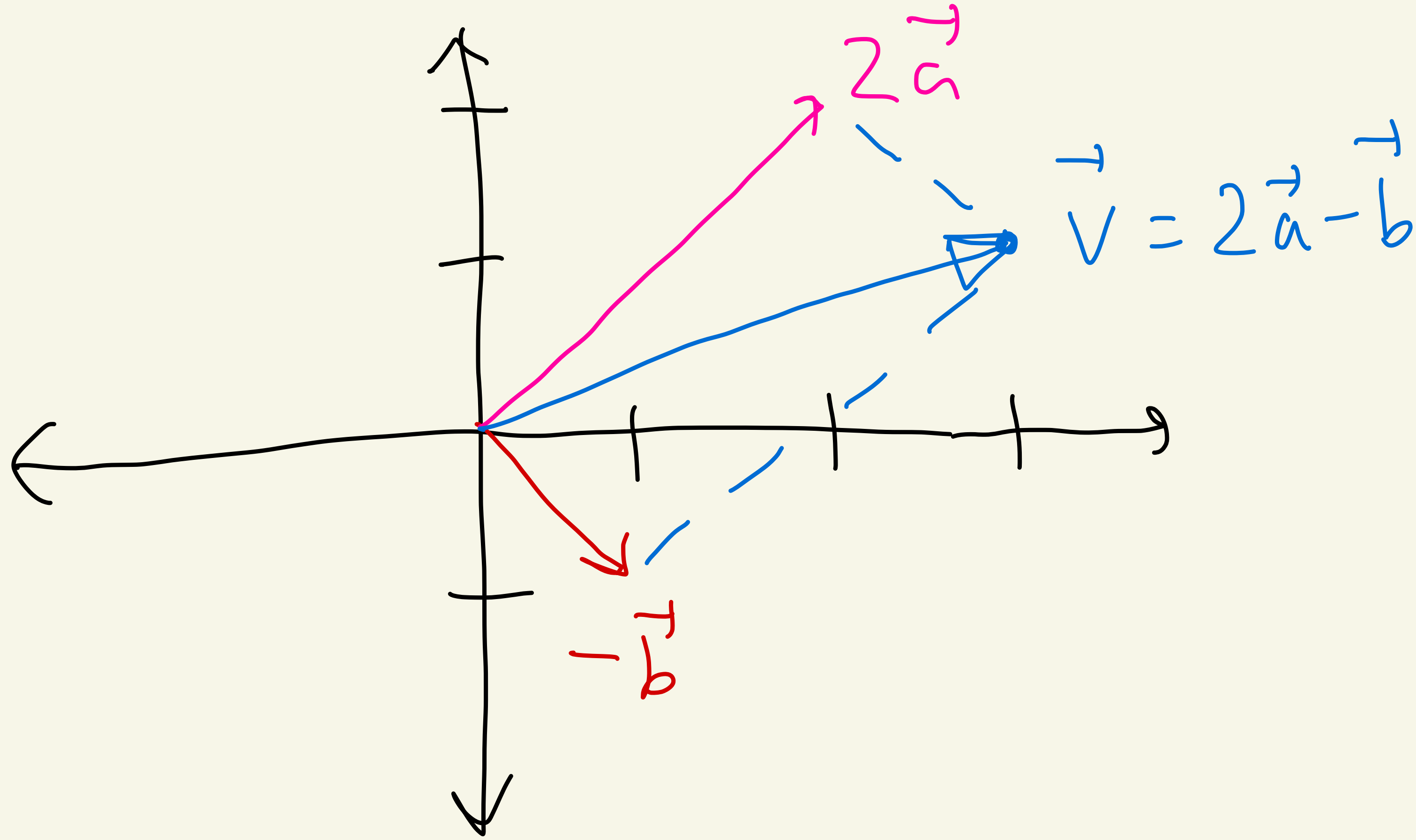
So, $[\vec{v}]_\beta = \underbrace{\langle 2, -1\rangle}_{\text{coordinates of } \vec{v} \text{ w/ respect to } \beta}$

Let's draw a picture of the coordinate system β and $\vec{v} = \langle 3, 1 \rangle$.

$\boxed{\vec{a} = \langle 1, 1 \rangle \quad \vec{b} = \langle -1, 1 \rangle}$



Simplified picture for $\vec{v}$



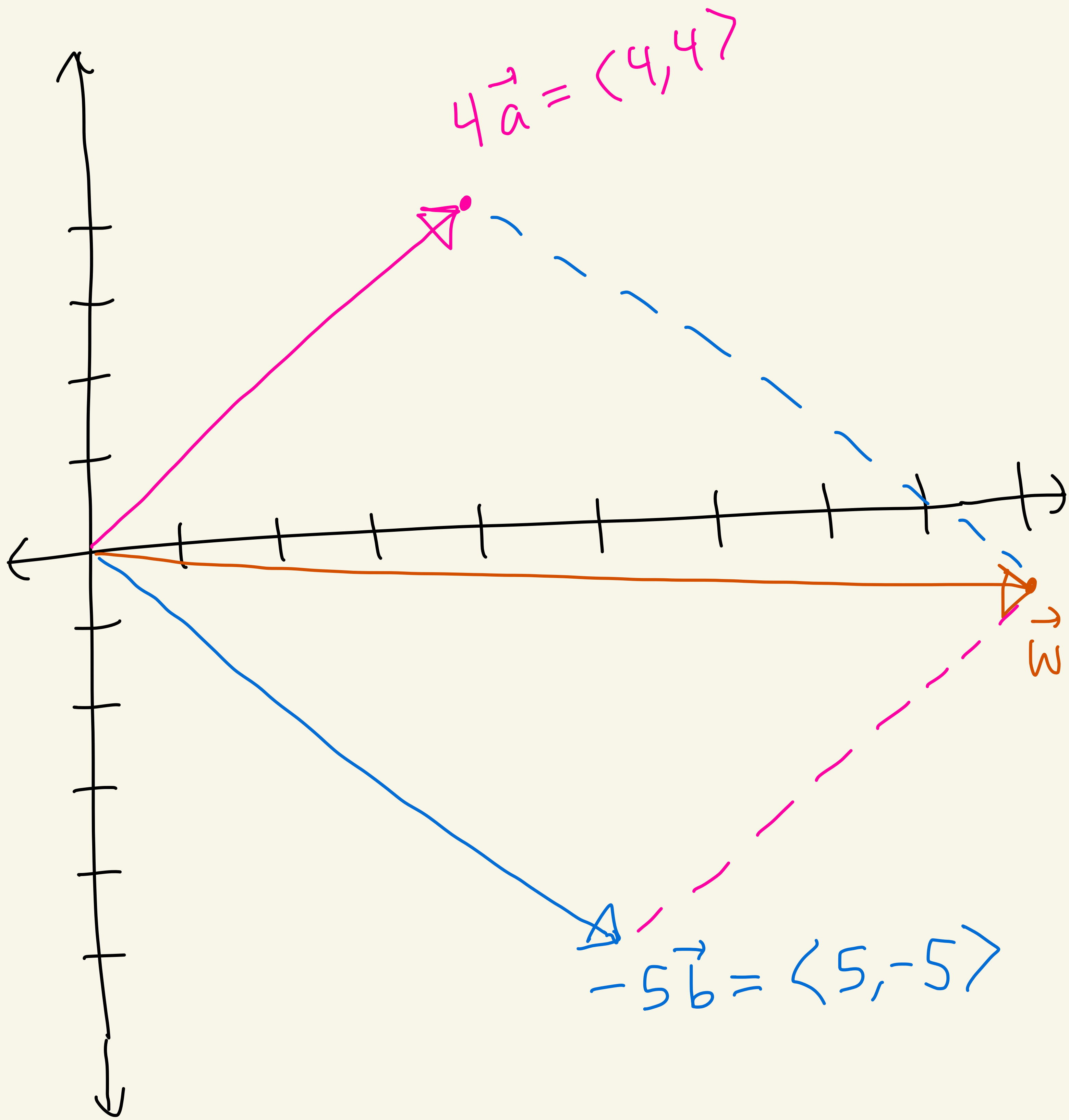
Q: Suppose you know that $\underbrace{[\vec{w}]_\beta}_{\vec{w}\text{'s } \beta\text{-coordinates}} = \langle 4, -5 \rangle$. What is $\vec{w}$?

$\beta = [\vec{a}, \vec{b}]$

This tell's us $\vec{w} = 4\vec{a} - 5\vec{b}$

Thus, $\vec{w} = 4\langle 1,1\rangle - 5\langle -1,1\rangle$

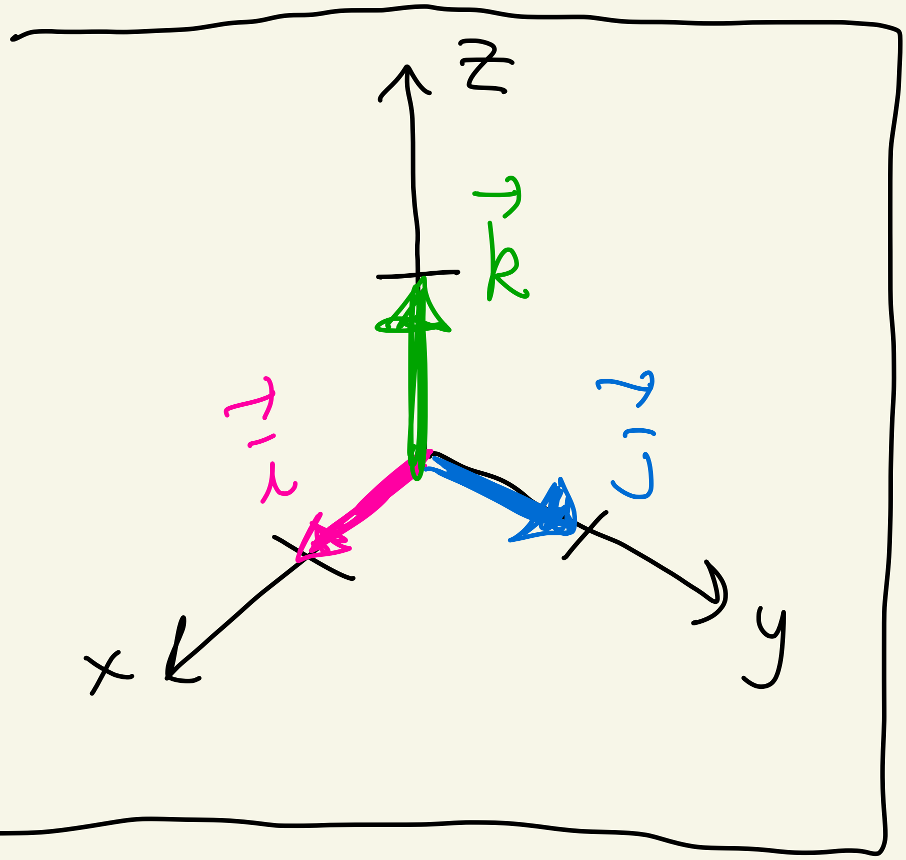
So, $\vec{w} = \langle 9,-1\rangle$.



Ex: In $\mathbb{R}^3$, let

$\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$,

$\vec{k} = \langle 0, 0, 1 \rangle$.



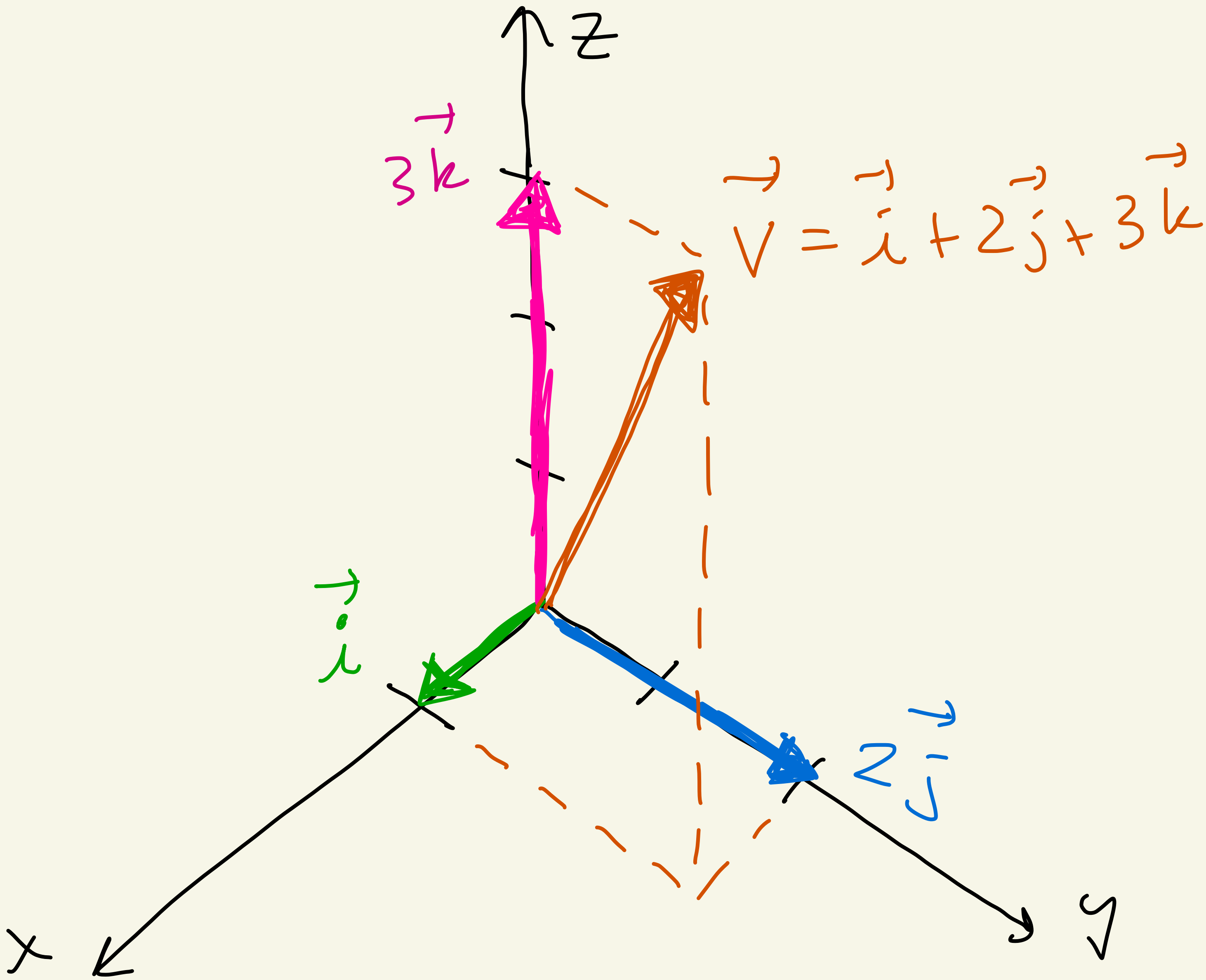
In the HW you will show these vectors are linearly independent.

So, $\beta = [\vec{i}, \vec{j}, \vec{k}]$ is a coordinate system or basis for $\mathbb{R}^3$. This is called the <u>standard basis</u> for $\mathbb{R}^3$

Let $\vec{v} = \langle 1, 2, 3 \rangle$.

Then,

$$\vec{v} = \langle 1,0,0 \rangle + \langle 0,2,0 \rangle + \langle 0,0,3 \rangle$$
$$= 1\langle 1,0,0 \rangle + 2\langle 0,1,0 \rangle + 3\langle 0,0,1 \rangle$$
$$= 1 \cdot \vec{i} + 2 \cdot \vec{j} + 3 \cdot \vec{k}$$



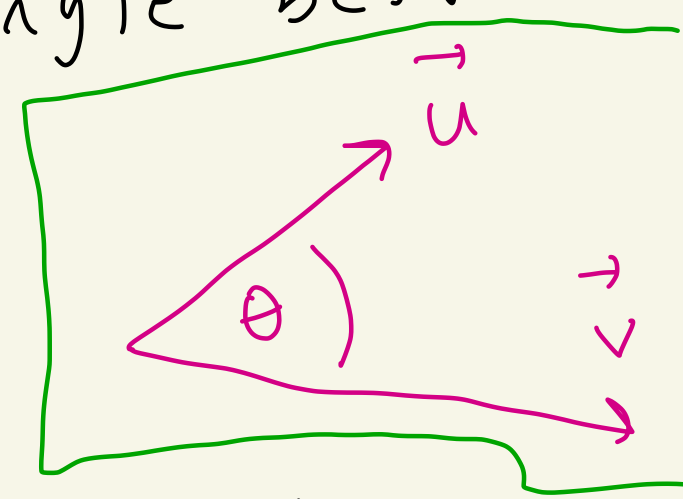
So for above $[\vec{v}]_\beta = \langle 1, 2, 3 \rangle$

since $\vec{v} = 1 \cdot \vec{i} + 2\vec{j} + 3\vec{k}$

Recall that $\vec{u}$ and $\vec{v}$ are in $\mathbb{R}^2$ or $\mathbb{R}^3$ then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

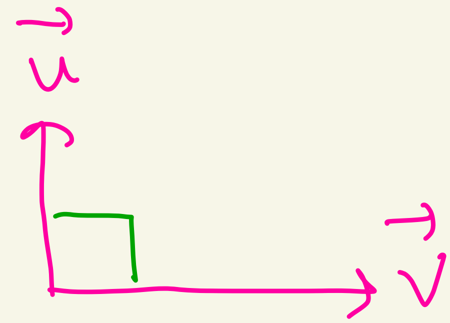


Therefore, $\theta = 90^\circ$ exactly when

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \underbrace{\cos(90^\circ)}_{0} = 0$$

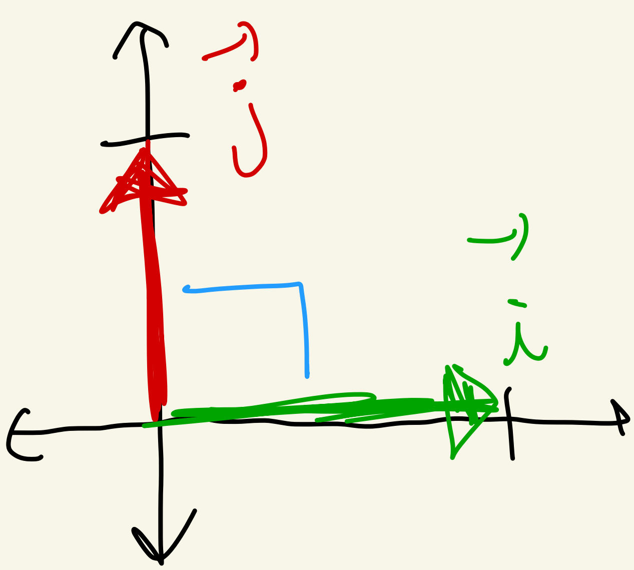
Def: Given two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ we say they are **orthogonal** if

$$\vec{u} \cdot \vec{v} = 0$$



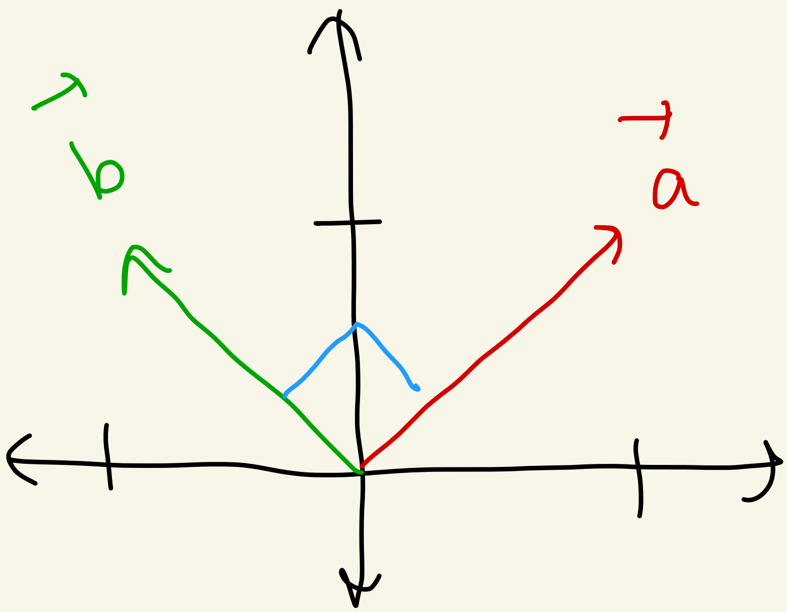
Ex: In $\mathbb{R}^2$,

$$\vec{i} \cdot \vec{j} = \langle 1, 0 \rangle \cdot \langle 0, 1 \rangle = 1 \cdot 0 + 0 \cdot 1$$
$$= 0$$



So, $\vec{i}, \vec{j}$ are orthogonal.

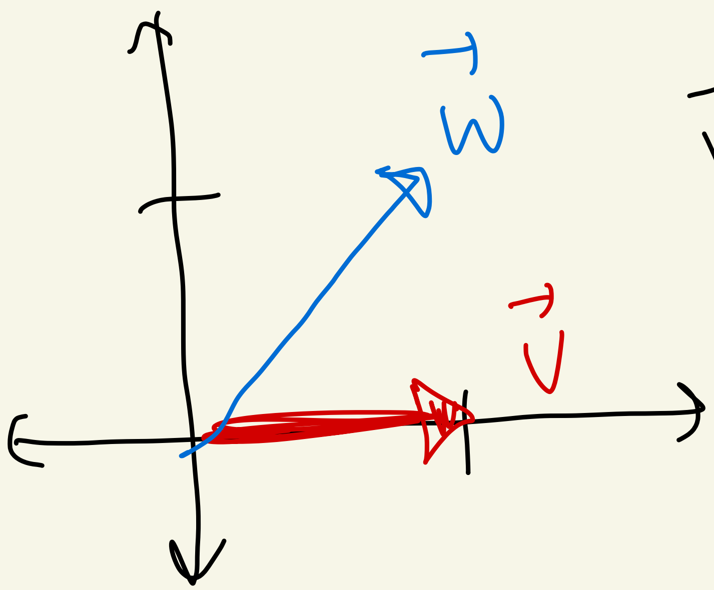
Ex: In $\mathbb{R}^2$, $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$



$$\vec{a} \cdot \vec{b} = (1)(-1) + (1)(1)$$
$$= 0$$

So, $\vec{a}$, $\vec{b}$ are orthogonal.

Ex: $\vec{v} = \langle 1, 0 \rangle$, $\vec{w} = \langle 1, 1 \rangle$



$$\vec{v} \cdot \vec{w} = (1)(1) + (0)(1)$$
$$= 1 \neq 0$$

$\vec{v}$, $\vec{w}$ are not orthogonal.