Math 2550-04

10/30/24

Topic 7 - Subspaces of $\mathbb{R}^n$

<u>Def:</u> Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ be r vectors in $\mathbb{R}^n$. The set of all linear combinations

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r$$

of these vectors is called the <u>subspace</u> spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$. We denote it by

$$\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r)$$

$$= \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r \;\middle|\; c_1, c_2, \ldots, c_r \in \mathbb{R} \right\}$$

Call this subspace W.

If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are linearly independent

then we say that the <u>dimension</u> of W is r and write $\dim(W) = r$.
And we call $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r]$ a <u>basis</u> for W.

<u>Ex:</u> In $\mathbb{R}^2$, let $\vec{v} = \langle 1, 2 \rangle$.

Let

$$W = \operatorname{span}(\vec{v}) = \{ c\vec{v} \mid c \in \mathbb{R} \}$$

For example some vectors in W are:

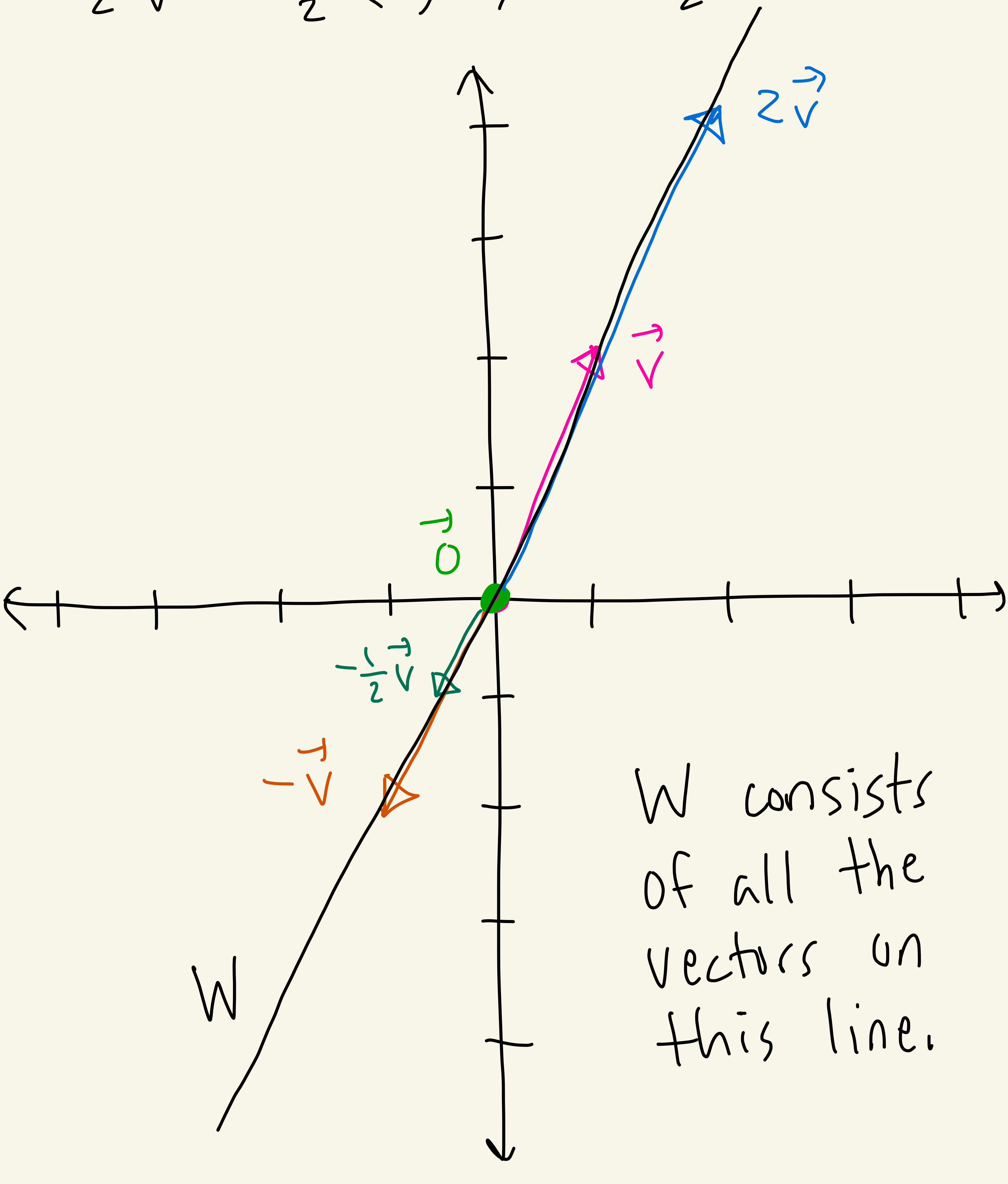
$$0 \cdot \vec{v} = 0 \langle 1, 2 \rangle = \langle 0, 0 \rangle$$

$$1 \cdot \vec{v} = 1 \langle 1, 2 \rangle = \langle 1, 2 \rangle$$

$$-1 \cdot \vec{v} = -1 \langle 1, 2 \rangle = \langle -1, -2 \rangle$$

$$2 \cdot \vec{v} = 2 \langle 1, 2 \rangle = \langle 2, 4 \rangle$$

$$-\frac{1}{2}\vec{v} = -\frac{1}{2}\langle 1, 2\rangle = \langle -\frac{1}{2}, -1\rangle$$



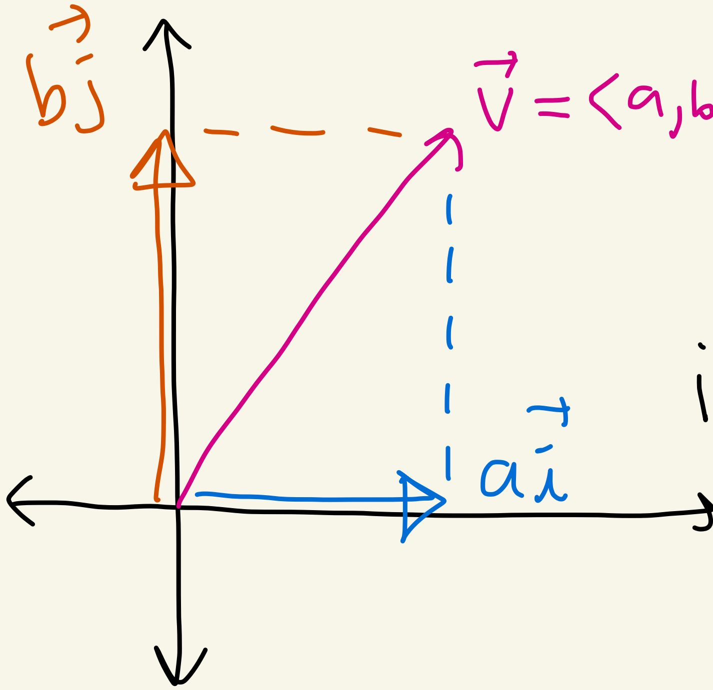
W consists of all the vectors on this line.

Since $\vec{v} = \langle 1,2 \rangle$ is not the zero vector we get that $\beta = [\vec{v}]$ is a linearly independent set and so its a basis for W.

Since β has only one vector the dimension of W is $\dim(W) = 1$

<u>Ex:</u> In $\mathbb{R}^2$, let $\vec{i} = \langle 1,0 \rangle$, $\vec{j} = \langle 0,1 \rangle$. We already know that $\beta = [\vec{i}, \vec{j}]$ is a basis for all of $\mathbb{R}^2$, that is, any vector $\vec{v} = \langle a,b \rangle$ in $\mathbb{R}^2$ is in the span of $\vec{i}, \vec{j}$ because

$$\vec{v} = \langle a,b \rangle = a\vec{i} + b\vec{j}$$



$\vec{v} = \langle a, b \rangle = a\vec{i} + b\vec{j}$

Here $\mathbb{R}^2 = \text{span}(\vec{i}, \vec{j})$ is a subspace of itself.

We get

$\dim(\mathbb{R}^2) = 2$

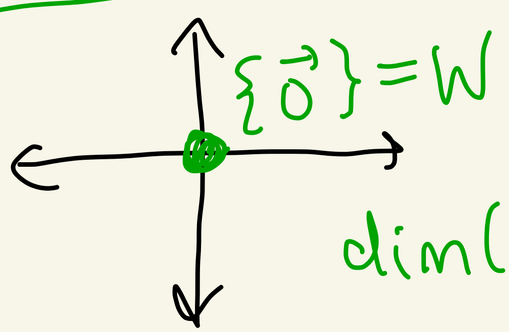
since a basis β has 2 vectors in it.

<u>Def:</u> In $\mathbb{R}^n$, let

$W = \text{span}(\vec{0}) = \{\vec{0}\}$

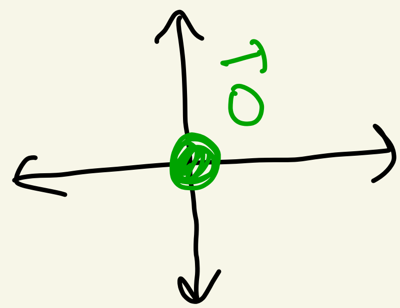
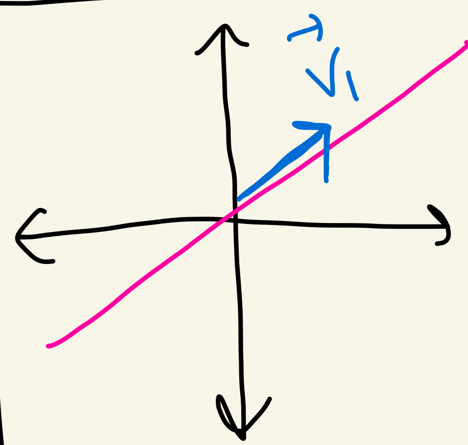
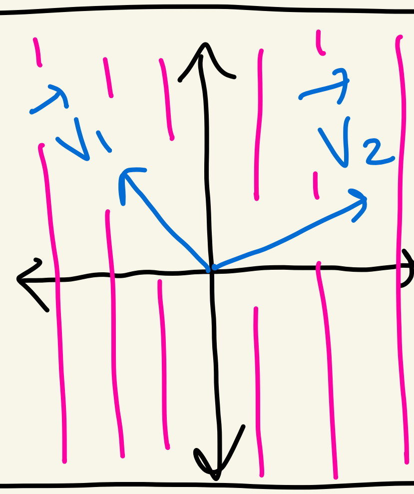
W has no basis, but we define its dimension to be 0.

In $\mathbb{R}^2$:

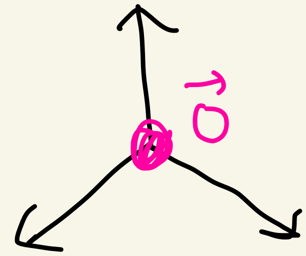
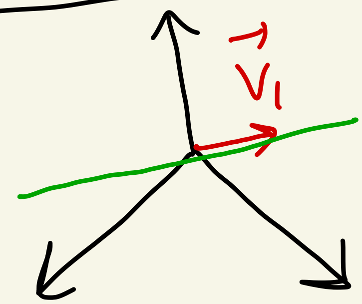
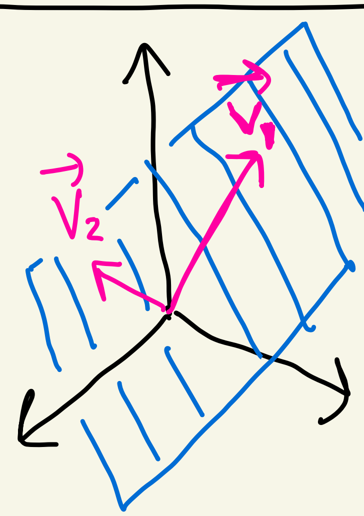
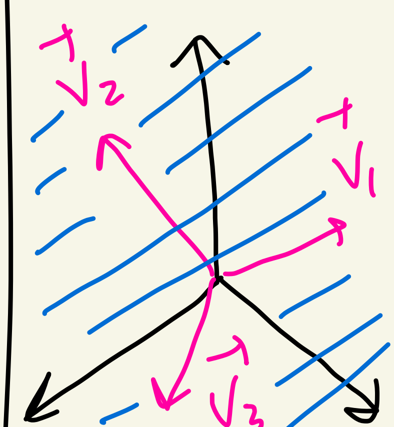


$W = \{\vec{0}\}$ is called the <u>trivial</u> <u>subspace</u> of $\mathbb{R}^n$

All subspaces of $\mathbb{R}^2$

dimension m	basis of m linearly independent vectors	picture of span of basis	description
0	no basis		point at origin
1	$\vec{v}_1 \neq \vec{0}$		line through origin
2	$\vec{v}_1, \vec{v}_2$		the entire plane $\mathbb{R}^2$

Subspaces in $\mathbb{R}^3$

dimension m	basis of m linearly independent vectors	picture of span of basis	description
0	no basis	 	point at origin
1	$\vec{v}_1$	 	line through origin
2	$\vec{v}_1, \vec{v}_2$	 	a plane through the origin that $\vec{v}_1$ $\vec{v}_2$ lie on
3	$\vec{v}_1, \vec{v}_2, \vec{v}_3$	 	all of $\mathbb{R}^3$

Homogeneous subspace theorem

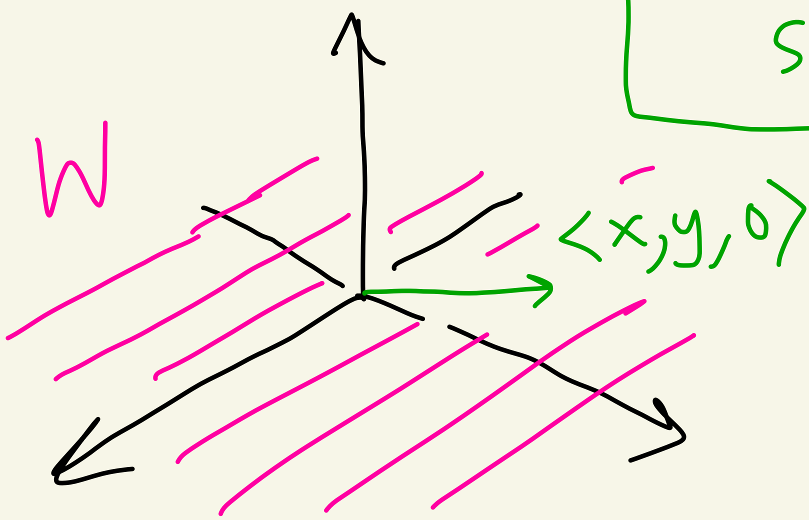
Let W be a subset of $\mathbb{R}^n$.
Then W is a subspace if and only if W consists of all vectors $\vec{v} = \langle x_1, x_2, \ldots, x_n \rangle$ that solve a homogeneous system of linear equations

$$
\boxed{
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
}
$$

homogeneous means $= 0$ on all equations

<u>Ex:</u> In $\mathbb{R}^3$, let

$$W = \{ \langle x, y, z \rangle \mid \underbrace{z = 0}_{\text{homogeneous system}} \}$$



W consists of all the vectors in the xy-plane

Let's make a basis that spans W

Let $\vec{v}$ be in W.

Then, $\vec{v} = \langle x, y, 0 \rangle$.

$\boxed{z = 0}$

So,

$$\vec{v} = \langle x, y, 0 \rangle = \langle x, 0, 0 \rangle + \langle 0, y, 0 \rangle$$

$$= x \langle 1, 0, 0 \rangle + y \langle 0, 1, 0 \rangle$$

$$= x \vec{i} + y \vec{j}$$

So, $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$

span W.

Are $\vec{i}, \vec{j}$ linearly independent?

Consider

$$c_1 \vec{i} + c_2 \vec{j} = \vec{0}$$

We get

$$c_1 \langle 1, 0, 0 \rangle + c_2 \langle 0, 1, 0 \rangle = \langle 0, 0, 0 \rangle$$

giving

$$\langle c_1, c_2, 0 \rangle = \langle 0, 0, 0 \rangle$$

This gives $c_1 = 0$, $c_2 = 0$.

Since the only solution to

$$c_1 \vec{i} + c_2 \vec{j} = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$. We get $\vec{i}, \vec{j}$ are linearly independent.

So, $\beta = [\vec{i}, \vec{j}]$ is a basis for W and $\dim(W) = 2$ since there are two vectors in the basis.