Math 2550-04

11/6/24

Last time, we turned a system into

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

and did

$$A^{-1}A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

and got

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}}_{A^{-1}}\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

So,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (-1)(4)+(0)(-1)+(1)(3) \\ (0)(4)+(-1)(-1)+(1)(3) \\ (2)(4)+(3)(-1)+(-4)(3) \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 4 \\ -7 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \text{Answer} \\ x_1 = -1 \\ x_2 = 4 \\ x_3 = -7 \end{array}}$$

HW 6
1(c)

Show the vectors are linearly dependent and write one as a linear combo of the others.

$\vec{v} = \langle 2, -1, 3 \rangle$

$\vec{u} = \langle 4, 1, 2 \rangle$

$\vec{w} = \langle 8, -1, 8 \rangle$

Want to solve

$$c_1 \vec{v} + c_2 \vec{u} + c_3 \vec{w} = \vec{0}$$

We get

$$c_1 \langle 2, -1, 3 \rangle + c_2 \langle 4, 1, 2 \rangle + c_3 \langle 8, -1, 8 \rangle = \langle 0, 0, 0 \rangle$$

We get

$$\langle 2c_1, -c_1, 3c_1\rangle + \langle 4c_2, c_2, 2c_2\rangle + \langle 8c_3, -c_3, 8c_3\rangle = \langle 0,0,0\rangle$$

So,

$$\langle \underbrace{2c_1+4c_2+8c_3}, \underbrace{-c_1+c_2-c_3}, \underbrace{3c_1+2c_2+8c_3}\rangle = \langle 0,0,0\rangle$$

So we get

$$\boxed{\begin{aligned} 2c_1+4c_2+8c_3 &= 0 \\ -c_1+c_2-c_3 &= 0 \\ 3c_1+2c_2+8c_3 &= 0 \end{aligned}}$$

Solve:

$$\left(\begin{array}{ccc|c} 2 & 4 & 8 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & 2 & 8 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & 4 & 8 & 0 \\ 3 & 2 & 8 & 0 \end{array}\right)$$

$$\xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 4 & 8 & 0 \\ 3 & 2 & 8 & 0 \end{array}\right)$$

$$\xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 6 & 6 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right)$$

$$\xrightarrow{1/6\, R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right)$$

$$\xrightarrow{-5R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Gives:

$$\boxed{\begin{aligned} c_1 - c_2 + c_3 &= 0 \quad (1) \\ c_2 + c_3 &= 0 \quad (2) \\ 0 &= 0 \end{aligned}}$$

leading: c_1, c_2

free: c_3

Solving:

$c_3 = t$

(2) $c_2 = -c_3 = -t$

(1) $c_1 = c_2 - c_3 = -t - t = -2t$

We get:

$$\underbrace{(-2t)\vec{v} + (-t)\vec{u} + (t)\vec{w} = \vec{0}}_{c_1\vec{v} + c_2\vec{u} + c_3\vec{w} = \vec{0}}$$

Plug in $t=1$ to get:

$$\boxed{-2\vec{v} - \vec{u} + \vec{w} = \vec{0}}$$

shows: $\vec{u}, \vec{v}, \vec{w}$ lin. dep.

And

$$\boxed{\vec{w} = 2\vec{v} + \vec{u}}$$

writing one as a lin. combo of the others

HW 6

② In $\mathbb{R}^2$, $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$

(a) Show $\vec{a}, \vec{b}$ are linearly independent.

Consider

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

We get

$$c_1 \langle 1, 1 \rangle + c_2 \langle -1, 1 \rangle = \langle 0, 0 \rangle$$

$$\langle c_1, c_1 \rangle + \langle -c_2, c_2 \rangle = \langle 0, 0 \rangle$$

$$\langle c_1 - c_2, c_1 + c_2 \rangle = \langle 0, 0 \rangle$$

We get
$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{-R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

We get

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \\ c_2 &= 0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} &c_2 = 0 \\ &c_1 = c_2 = 0 \end{aligned}}$$

So, the only sols to

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

are $c_1 = 0$, $c_2 = 0$. Thus, $\vec{a}$ and $\vec{b}$ are linearly independent.

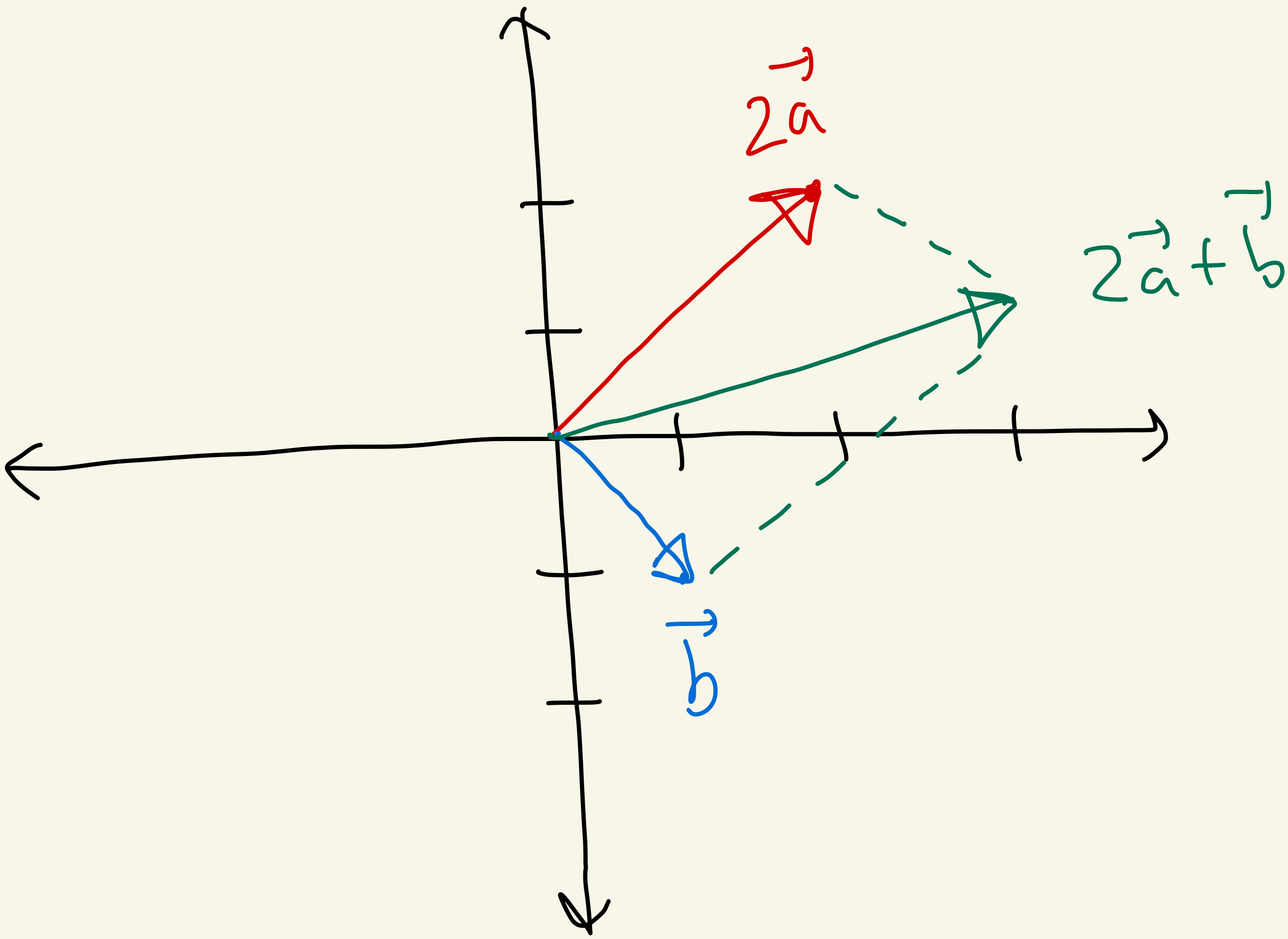
Since we have 2 lin. ind. vectors we get a basis for $\mathbb{R}^2$.

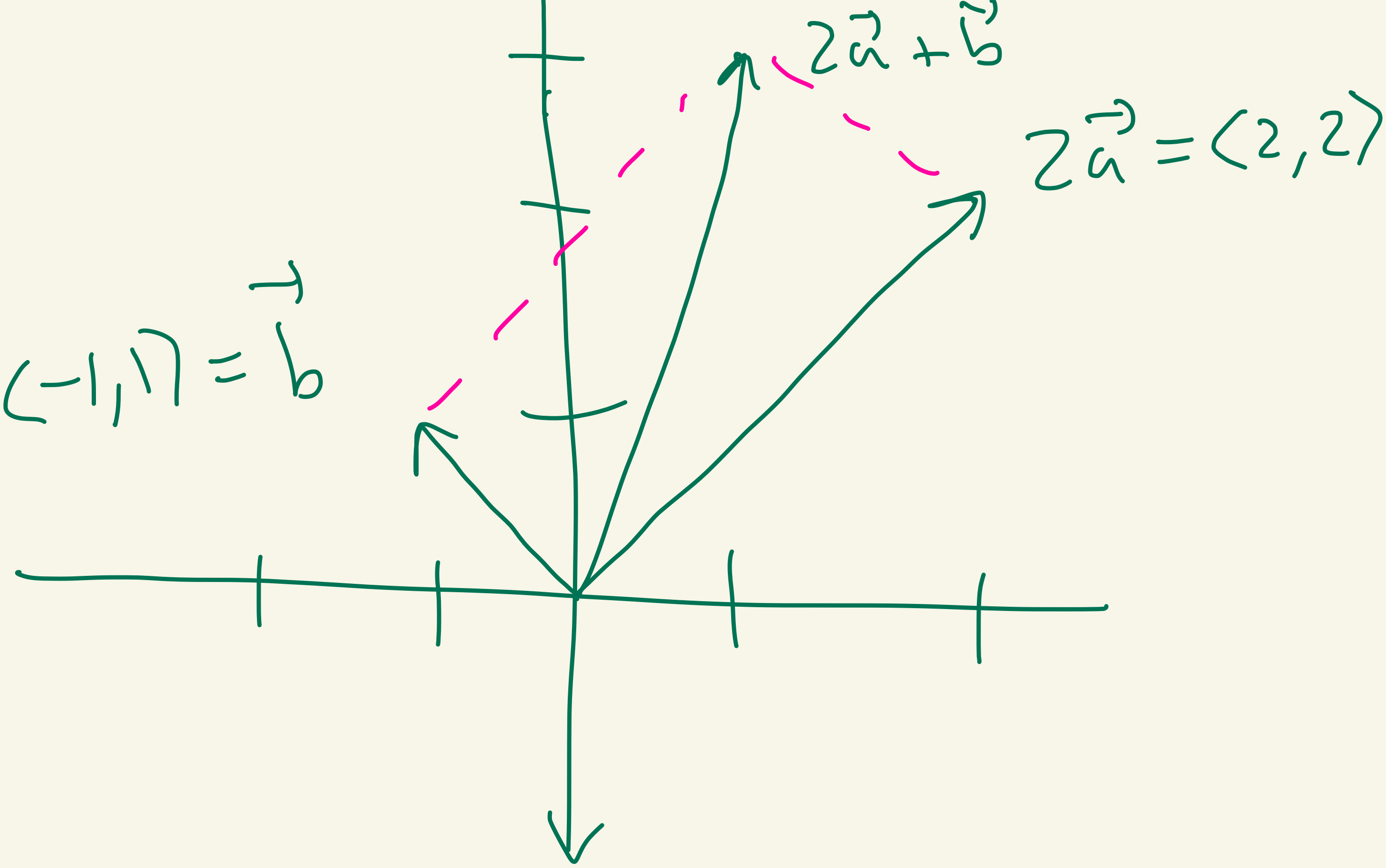
② (c) Draw $2\vec{a}$, $\vec{b}$, $2\vec{a}+\vec{b}$ and the parallelogram they make.

$2\vec{a} = 2\langle 1,1\rangle = \langle 2,2\rangle$

$\vec{b} = \langle 1,-1\rangle$

$2\vec{a}+\vec{b} = \langle 3,1\rangle$





② (g) Show $\beta = [\vec{a}, \vec{b}]$ is an orthogonal basis, but not orthonormal.

<u>orthogonal</u>

$$\left.\begin{aligned} \vec{a}\cdot\vec{b} &= \langle 1,1\rangle \cdot \langle -1,1\rangle \\ &= (1)(-1)+(1)(1) = 0 \end{aligned}\right] \checkmark$$

not orthonormal

$$\|\vec{a}\| = \sqrt{1^2+1^2} = \sqrt{2} \neq 1$$
$$\|\vec{b}\| = \sqrt{(-1)^2+1^2} = \sqrt{2} \neq 1$$

since not all 1 not orthonormal

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② (h) Use the coordinate dot product theorem to find $[\vec{v}]_\beta$ where $\vec{v} = \langle 10, \frac{1}{2} \rangle$.

Recall: $\beta = [\vec{a}, \vec{b}]$, $\vec{a} = \langle 1,1 \rangle$
$\vec{b} = \langle -1,1 \rangle$
is an orthogonal basis

The coordinate dot product theorem

tells us that

$$\vec{v} = \left(\frac{\vec{v}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a} + \left(\frac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

$$= \left(\frac{\langle 10, \frac{1}{2}\rangle \cdot \langle 1, 1\rangle}{(\sqrt{2})^2}\right)\vec{a} + \left(\frac{\langle 10, \frac{1}{2}\rangle \cdot \langle -1, 1\rangle}{(\sqrt{2})^2}\right)\vec{b}$$

$$= \left(\frac{10 + 1/2}{2}\right)\vec{a} + \left(\frac{-10 + 1/2}{2}\right)\vec{b}$$

$$= \frac{21}{4}\vec{a} - \frac{19}{4}\vec{b}$$

So, $\vec{v} = \frac{21}{4}\vec{a} - \frac{19}{4}\vec{b}$

Thus, $\boxed{[\vec{v}]_\beta = \left\langle \frac{21}{4}, \frac{-19}{4} \right\rangle}$

(2) (j) Suppose you know $[\vec{v}]_\beta = \langle 5, -4 \rangle$. What is $\vec{v}$?

$\beta = [\vec{a}, \vec{b}]$

$\vec{a} = \langle 1, 1 \rangle$

$\vec{b} = \langle -1, 1 \rangle$

This means that:

$$\begin{aligned}
\vec{v} &= 5\vec{a} - 4\vec{b} \\
&= 5\langle 1, 1 \rangle - 4\langle -1, 1 \rangle \\
&= \langle 5, 5 \rangle + \langle 4, -4 \rangle \\
&= \langle 5+4, 5-4 \rangle = \boxed{\langle 9, 1 \rangle}
\end{aligned}$$

Calculate $\det(A)$ where

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & -1 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= -0 + (-1)\underbrace{\begin{vmatrix} 1 & -2 \\ 4 & 0 \end{vmatrix}}_{\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & -1 & 0 \end{pmatrix}} - (-1)\underbrace{\begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix}}_{\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & -1 & 0 \end{pmatrix}}$$

$$= 0 - \left[(1)(0) - (-2)(4)\right] + \left[(1)(2) - (-2)(3)\right]$$

$= -8 + 8 = 0$

Since $\det(A) = 0$ we know A^{-1} does not exist.