Math 2550-04
12/2/24

Topic 9 – Matrices of linear transformations

<u>Ex:</u> Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 3x \\ 8x-y \end{pmatrix}$$

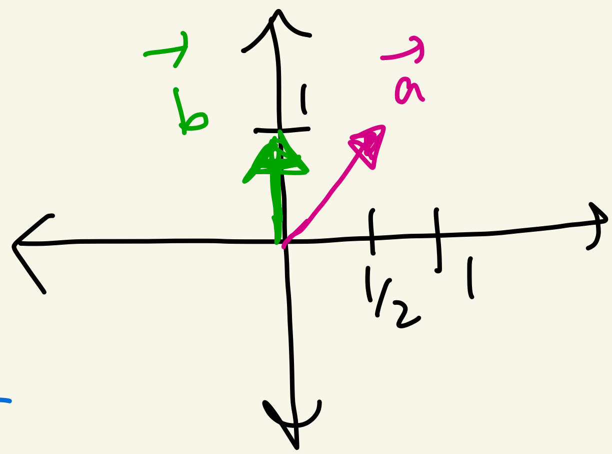
We saw that the eigenvalues of T are $\lambda = 3, -1$ with eigenvectors $\vec{a} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

That is, $T(\vec{a}) = 3\vec{a}$

$$T(\vec{b}) = -\vec{b}$$

Let $\beta = [\vec{a}, \vec{b}]$. Then you can show $\vec{a}, \vec{b}$ are lin. ind. so β is a basis/coordinate system for $\mathbb{R}^2$.

basis of T's eigenvectors



Idea: Given any other vector $\vec{v}$ we can decompose it in terms of the basis β like this

$$\vec{v} = c_1 \vec{a} + c_2 \vec{b}$$

Ex:

$\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\vec{v} = 2\vec{a} + 1 \cdot \vec{b}$

$\left[\vec{a} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$

$T(\vec{v})$

Then,

$$T(\vec{v}) = T(c_1\vec{a} + c_2\vec{b})$$
$$= A(c_1\vec{a} + c_2\vec{b})$$
$$= A(c_1\vec{a}) + A(c_2\vec{b})$$
$$= c_1 A\vec{a} + c_2 A\vec{b}$$
$$= c_1(3\vec{a}) + c_2(-\vec{b})$$
$$= (3c_1)\vec{a} + (-c_2)\vec{b}$$

$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

eigen-vector

$$= T\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$
$$= 6\vec{a} - \vec{b}$$
$$= 3(2\vec{a}) - (1 \cdot \vec{b})$$

So T turns β-coordinates

$[\vec{v}]_\beta = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ into β-coordinates

$[T(\vec{v})]_\beta = \begin{pmatrix} 3c_1 \\ -c_2 \end{pmatrix}$. The matrix that does this is: $\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

Another way to think of T:

$$\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{new matrix that computes } T} \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\beta\text{-coordinates input}} = \underbrace{\begin{pmatrix} 3c_1 \\ -c_2 \end{pmatrix}}_{T \text{ applied to the } \beta\text{-coordinates but outputed as } \beta\text{-coordinates}}$$

$$T(c_1\vec{a} + c_2\vec{b}) = (3c_1)\vec{a} + (-c_2)\vec{b}$$

Let $\vec{w} = -2\vec{a} + 6\vec{b}$ $\leftarrow$ $\boxed{-2\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \vec{w}}$

Do this:

$$\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{new matrix for } T} \underbrace{\begin{pmatrix} -2 \\ 6 \end{pmatrix}}_{\vec{w}\text{'s } \beta\text{-coordinates}} = \begin{pmatrix} 3(-2) \\ -(6) \end{pmatrix} = \underbrace{\begin{pmatrix} -6 \\ -6 \end{pmatrix}}_{T(\vec{w})\text{'s } \beta\text{-coordinates}}$$

$$T(\vec{w}) = -6\vec{a} - 6\vec{b}$$
$$= -6\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} - 6\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -12 \end{pmatrix}$$

<u>Def:</u> Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ be a basis/coordinate system for $\mathbb{R}^n$. The matrix

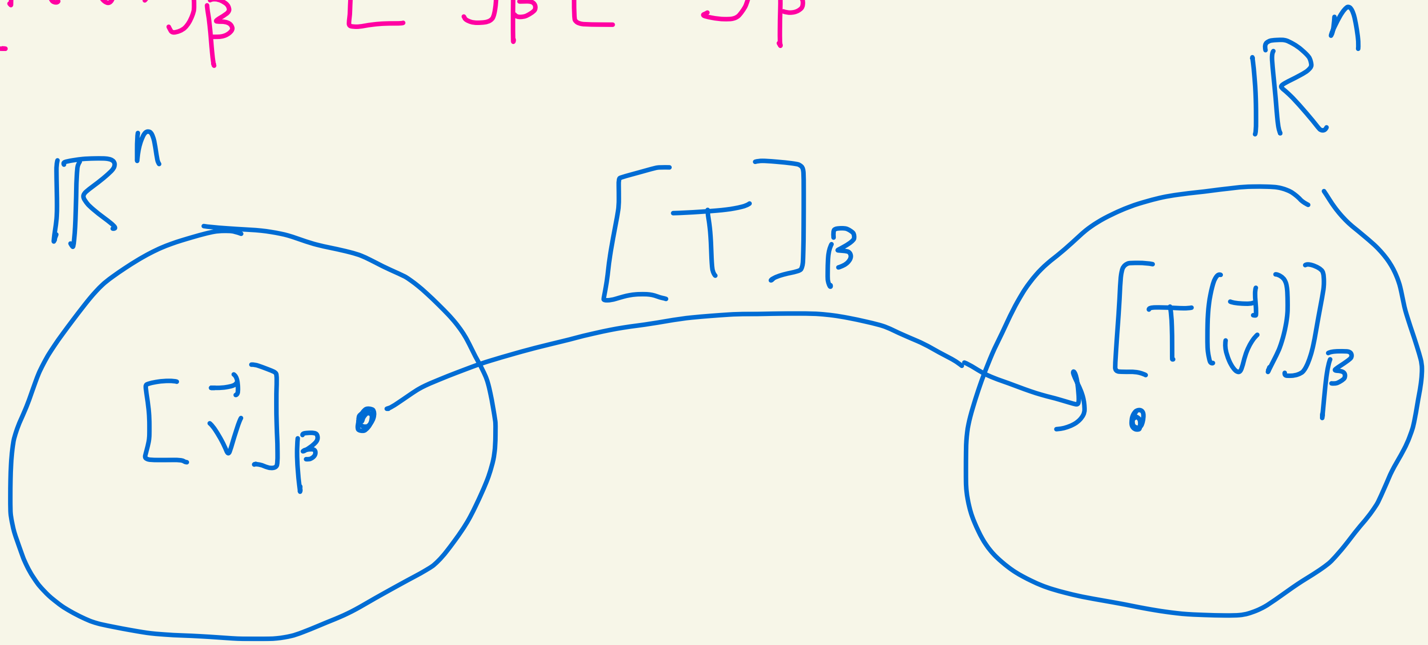
$$[T]_\beta = \begin{array}{c} \\ \\ \end{array}\left(\begin{array}{c|c|c|c} [T(\vec{v}_1)]_\beta & [T(\vec{v}_2)]_\beta & \cdots & [T(\vec{v}_n)]_\beta \\ \uparrow & \uparrow & & \uparrow \\ \text{column } 1 & \text{column } 2 & & \text{column } n \end{array} \right)$$

is called the <u>matrix for T</u>

with respect to β.

The matrix does this:

$$[T(\vec{v})]_\beta = [T]_\beta [\vec{v}]_\beta$$

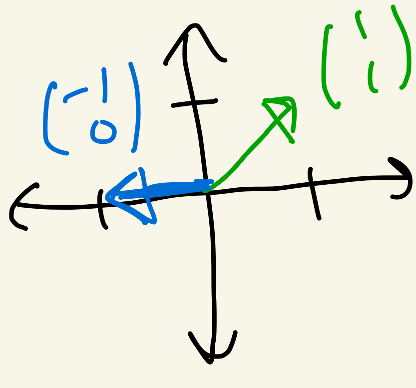


Ex: $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$$

Pick the basis

$$\beta = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]$$



Let's find $[T]_\beta$.

$$\underbrace{T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1-1\\1+1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}} = 2\cdot\begin{pmatrix}1\\1\end{pmatrix} + 2\begin{pmatrix}-1\\0\end{pmatrix}$$

$$T\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}-1-0\\-1+0\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix} = -1\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}-1\\0\end{pmatrix}$$

plug β into T

write the answers in β-coordinates

$$[T]_\beta = \left(\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_\beta \middle| \left[T\begin{pmatrix}-1\\0\end{pmatrix}\right]_\beta \right)$$

$$= \underbrace{\begin{pmatrix}2 & -1\\2 & 0\end{pmatrix}}$$

this matrix will compute T but it wants β-coordinates as input and gives β-coordinates as output

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$$

Let $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Then, $T(\vec{v}) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let's compute T using $[T]_\beta = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}$

We need $\vec{v}$'s β-coordinates.

$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\beta = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right]$$

So, $[\vec{v}]_\beta = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

And,

$$[T]_\beta [\vec{v}]_\beta = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 0 - 1(-1) \\ 2 \cdot 0 + 0(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = [T(\vec{v})]_\beta$$

So,

$$T(\vec{v}) = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$