Math 2550-04
8/28/24

Topic 1 continued...

Def: Let

$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ and

$\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle$

be in $\mathbb{R}^n$.

Define the dot product to be

$$\vec{v} \cdot \vec{w} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Note: The dot product is a number

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 1, -1 \rangle$ and $\vec{w} = \langle 3, -4 \rangle$.

Then,

$$\vec{v} \cdot \vec{w} = \langle 1, -1 \rangle \cdot \langle 3, -4 \rangle$$

$$= (1)(3) + (-1)(-4)$$

$$= 7$$

In Calculus in $\mathbb{R}^2$ or $\mathbb{R}^3$,

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}$$

where θ is the angle between $\vec{v}$ and $\vec{w}$

<u>Ex:</u> In $\mathbb{R}^5$, let's calculate

$$\left\langle 0, 1, -2, 3, \tfrac{1}{2} \right\rangle \cdot \left\langle -1, \tfrac{1}{3}, 4, 2, 10 \right\rangle$$

$$= (0)(-1) + (1)\left(\tfrac{1}{3}\right) + (-2)(4) + (3)(2) + \left(\tfrac{1}{2}\right)(10)$$

$$= 0 + \tfrac{1}{3} - 8 + 6 + 5 = 3 + \tfrac{1}{3} = \tfrac{10}{3}$$

Properties of dot product

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$.
Let α be a $\underbrace{\text{scalar}}_{\text{number}}$ in $\mathbb{R}$.

Then:

① $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

② $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

③ $\alpha(\vec{u} \cdot \vec{v}) = (\alpha\vec{u}) \cdot \vec{v} = \vec{u} \cdot (\alpha\vec{v})$

$\boxed{\text{proof of (2) when } n=3}$:

Let $\vec{u}, \vec{v}, \vec{w}$ be in $\mathbb{R}^3$.

Then,

$$\vec{u} = \langle a, b, c \rangle$$
$$\vec{v} = \langle d, e, f \rangle$$
$$\vec{w} = \langle g, h, i \rangle$$

where $a, b, c, d, e, f, g, h, i$ are numbers.

Then,

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \langle a, b, c \rangle \cdot \left(\langle d, e, f \rangle + \langle g, h, i \rangle \right)$$

$$= \langle a, b, c \rangle \cdot \langle d+g, e+h, f+i \rangle$$

$$= a(d+g) + b(e+h) + c(f+i)$$

$$= ad + ag + be + bh + cf + ci$$

Also, we have

$\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

$$= \underbrace{\langle a,b,c\rangle\cdot\langle d,e,f\rangle}_{} + \underbrace{\langle a,b,c\rangle\cdot\langle g,h,i\rangle}_{}$$

$$= \underbrace{ad+be+cf}_{} + ag+bh+ci$$

$$= ad+ag+be+bh+cf+ci$$

EQUAL

Thus, $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

$\square$

HW 1 - Part 1

(10) List 3 elements from the set

$$S = \{ c_1 \langle 1,1,1 \rangle + c_2 \langle 0,0,5 \rangle \mid c_1, c_2 \in \mathbb{R} \}$$

<u>means:</u> S consists of all elements of the form $c_1 \langle 1,1,1 \rangle + c_2 \langle 0,0,5 \rangle$ where c_1, c_2 are real #s

When $c_1 = 1$ and $c_2 = 2$ we get

$c_1 \langle 1,1,1 \rangle + c_2 \langle 0,0,5 \rangle$

$= 1 \cdot \langle 1,1,1 \rangle + 2 \langle 0,0,5 \rangle$

$= \langle 1,1,1 \rangle + \langle 0,0,10 \rangle$

$= \langle 1, 1, 11 \rangle$

So, $\langle 1, 1, 11 \rangle$ is in S

When $c_1 = 0$ and $c_2 = 0$ we get

$0 \cdot \langle 1, 1, 1 \rangle + 0 \cdot \langle 0, 0, 5 \rangle$

$= \langle 0, 0, 0 \rangle + \langle 0, 0, 0 \rangle$

$= \langle 0, 0, 0 \rangle$

So, $\langle 0, 0, 0 \rangle$ is in S

When $c_1 = 1$ and $c_2 = 0$

$1 \cdot \langle 1, 1, 1 \rangle + 0 \cdot \langle 0, 0, 5 \rangle$

$= \langle 1, 1, 1 \rangle + \langle 0, 0, 0 \rangle$

$= \langle 1, 1, 1 \rangle$

So, $\langle 1, 1, 1 \rangle$ is in S

So,

$$S = \{\langle 1,1,11\rangle, \langle 0,0,0\rangle, \langle 1,1,1\rangle, \ldots\}$$

infinitely many more

Topic 2 - Matrices

Def: A **matrix** is a rectangular array of numbers. If M is a matrix and it has m rows and n columns then M is an $\underbrace{m \times n}_{\text{read: "m by n"}}$ matrix.

Abstractly we can write an $m \times n$ matrix like this:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Where a_{ij} is in row i and column j.

Ex:

$$M = \begin{pmatrix} 0 & 5 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

M is $\underset{\text{2 rows}}{2} \times \underset{\text{2 columns}}{2}$

$$\boxed{\begin{aligned} a_{11} &= 0 \\ a_{12} &= 5 \\ a_{21} &= 3 \\ a_{22} &= -1 \end{aligned}}$$

Ex:

$$A = \begin{pmatrix} 1 & 5 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix}$$

A is $\underset{\text{1 row}}{1} \times \underset{\text{3 columns}}{3}$

$$\boxed{\begin{aligned} a_{11} &= 1 \\ a_{12} &= 5 \\ a_{13} &= 3 \end{aligned}}$$

Ex:

$$B = \begin{pmatrix} 1 & 3 & -1 \\ 7 & 2 & 3 \\ 10 & 5 & 1 \\ 1/2 & 0 & 0 \end{pmatrix}$$

$a_{33} = 1$

$a_{42} = 0$

B is 4×3

You can think of a vector as a matrix, either as a row of column.

For example, $\vec{v} = \langle 5, 2, -3 \rangle$

You can think of $\vec{v}$ as:

$\begin{pmatrix} 5 & 2 & -3 \end{pmatrix}$

1×3 matrix

or

$\begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$

3×1 matrix

Def: Let A and B be $m \times n$ matrices [So, A and B have the same dimensions.]

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

Define $A+B$ to be

$$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

Define $A-B$ to be

$$A-B=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \cdots & a_{mn}-b_{mn} \end{pmatrix}$$

If α is a scalar, define

$$\alpha A=\begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$