Math 2550-04
9/11/24

Def: A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is a list of m linear equations:

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}} \quad (*)$$

where a_{ij} and b_i are constants.

The solution space to $(*)$ consists of all $(x_1, x_2, \ldots, x_n)$ that simultaneously solve all m equations. That is, the

Common solutions to all m equations.

The **augmented matrix** for $(*)$ is

$$\left(\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right)$$

x_1 column $\quad$ x_2 column $\quad$ x_n column $\quad$ represents equals sign

Ex:

$$\left.\begin{aligned} x+2y&=3\\ 4x+5y&=6 \end{aligned}\right]$$ system of $m=2$ linear equations, $n=2$ unknowns

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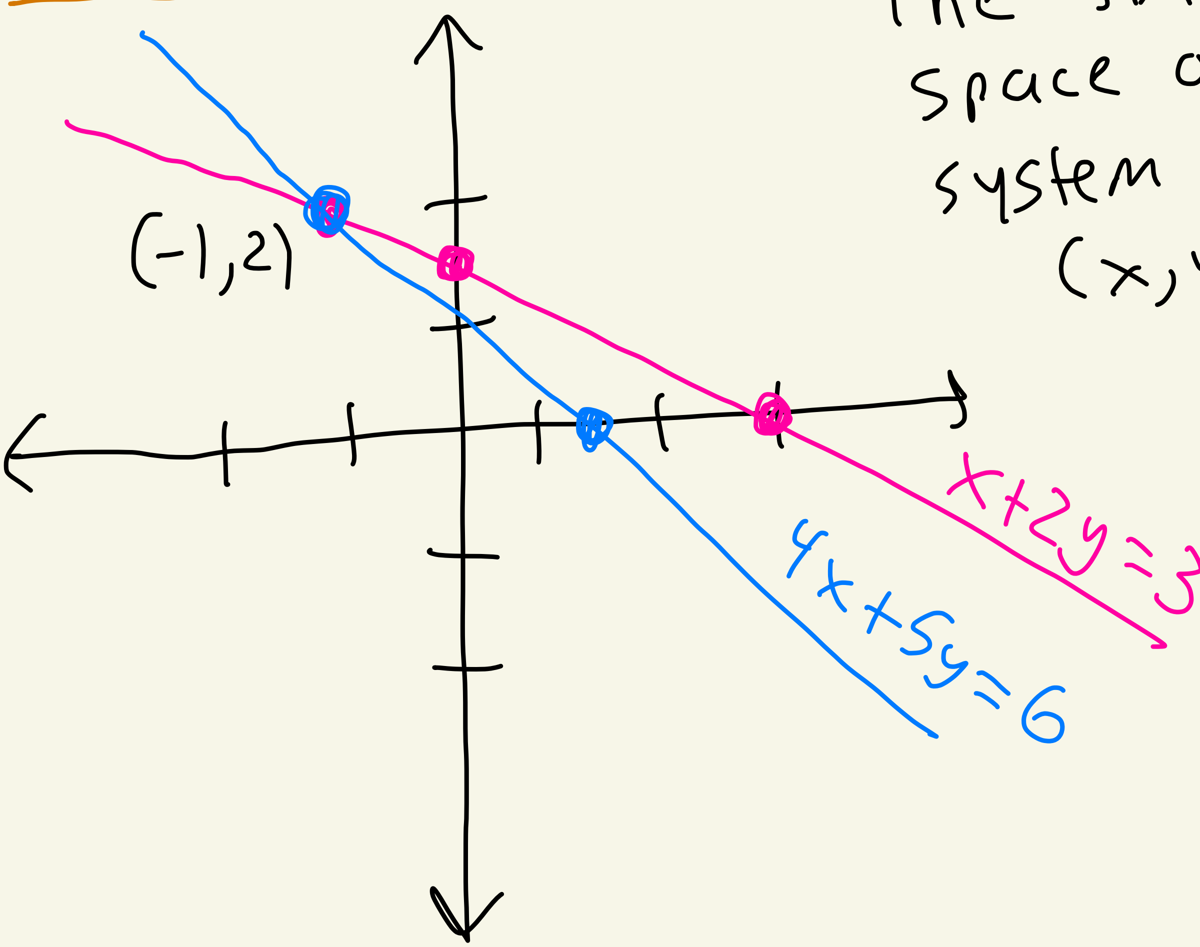
Augmented matrix

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right)$$



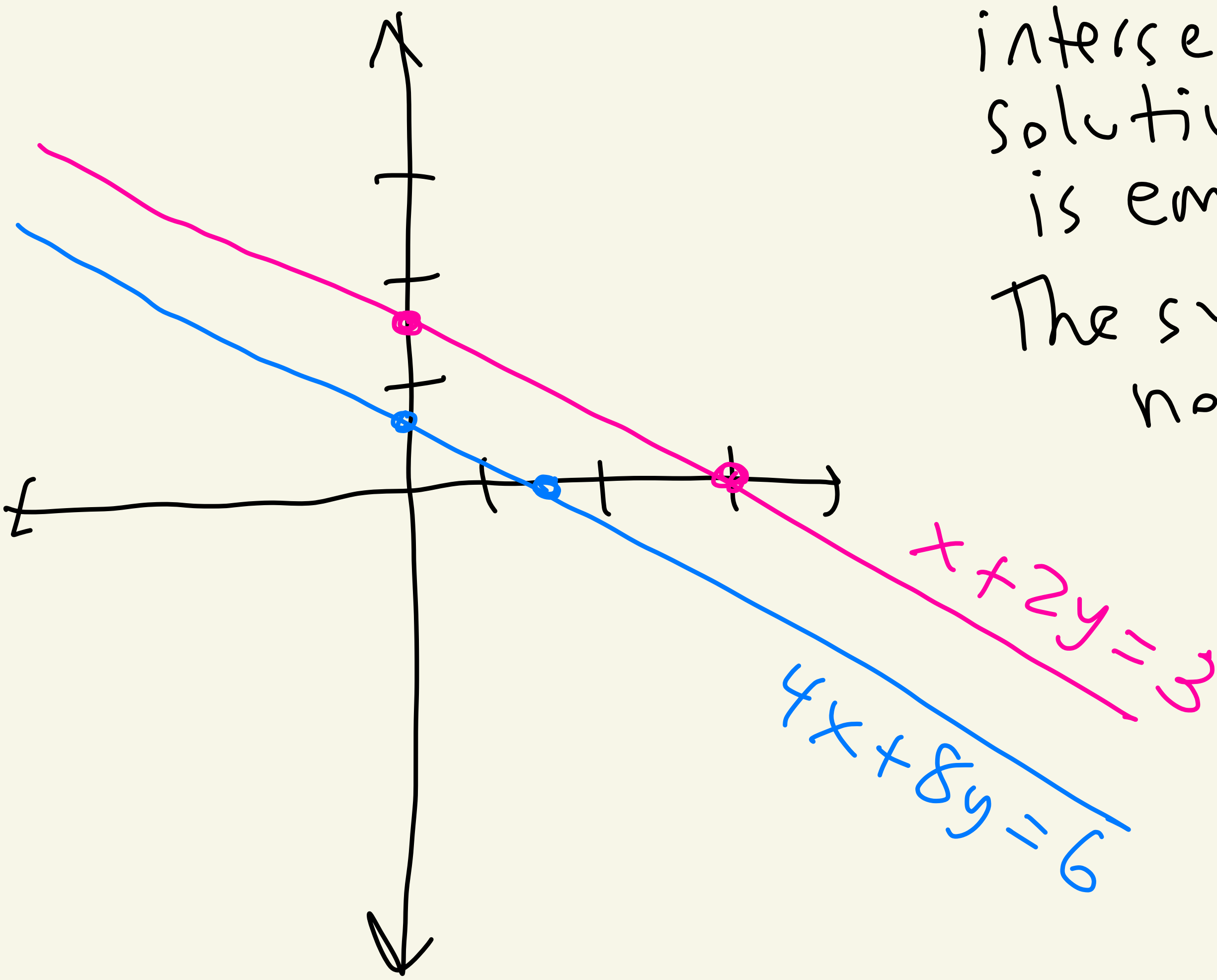
The solution space of the system is $(x,y)=(-1,2)$ or its the set $\{(-1,2)\}$

Ex: Consider

$$\left.\begin{aligned} x+2y&=3 \\ 4x+8y&=6 \end{aligned}\right]$$

$m=2$ equations
$n=2$ unknowns

Augmented matrix: $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 6 \end{array}\right)$



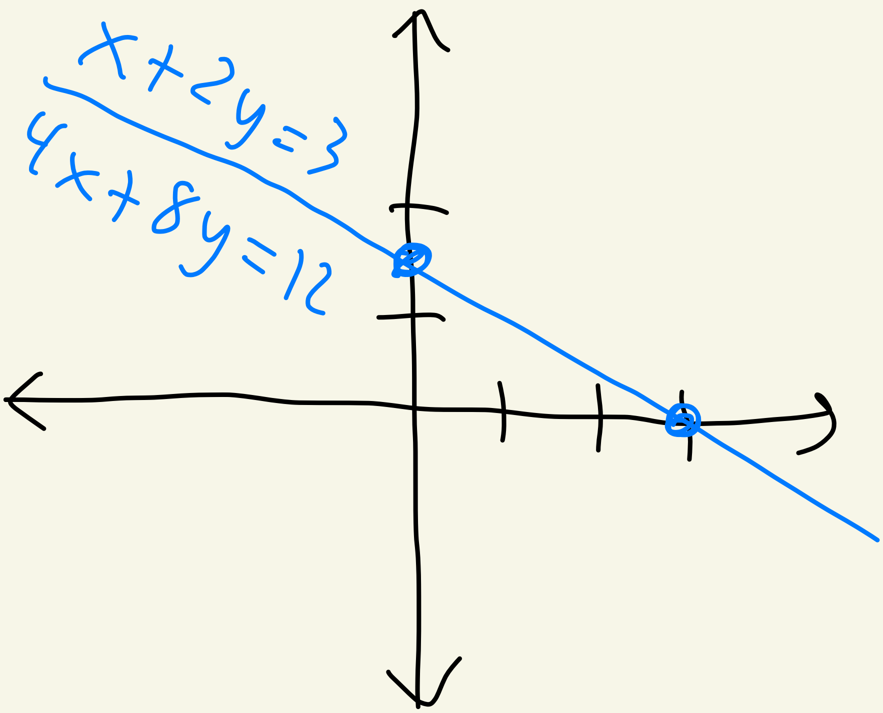
These lines don't intersect. The solution space is empty.

The system has no solutions.

Ex:

$$\left.\begin{aligned} x+2y&=3 \\ 4x+8y&=12 \end{aligned}\right] \quad \begin{aligned} m&=2 \text{ equations} \\ n&=2 \text{ unknowns} \end{aligned}$$

augmented matrix: $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 12 \end{array}\right)$



It's the same line twice. The solution space is infinite. It's all (x,y) that lie on the line.

Ex:

$$\boxed{\begin{aligned} x + y + 2z &= 9 \\ 2x \qquad - 3z &= 1 \\ -x + 6y - 5z &= 0 \end{aligned}}$$

$m = 3$ lin. eq.

$n = 3$ unknowns

Augmented matrix:

$$\begin{array}{ccc} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 0 & -3 & 1 \\ -1 & 6 & -5 & 0 \end{array}\right) \\ \begin{array}{ccc} x \text{ column} & y \text{ column} & z \text{ column} \end{array} \end{array}$$

Ex:

$$\boxed{\begin{aligned} 2y - 3w \phantom{{}+{}-2z} &= 5 \\ 4x + y + w - 2z &= 0 \\ x - y + w \phantom{{}-2z} &= 1 \end{aligned}}$$

$m = 3$ lin. eq.

$n = 4$ unknowns x, y, w, z

Augmented matrix:

$$\left(\begin{array}{cccc|c} 0 & 2 & -3 & 0 & 5 \\ 4 & 1 & 1 & -2 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{array}\right)$$

Def: Given a system of linear equations there are three operations that we call elementary row operations.

They are:

1. Multiply one of the rows/equations by a non-zero number
2. Interchange two rows/equations
3. Add a multiple of one row/equation to another row/equation

Ex: (multiplying a row/equation by non-zero #)

$\boxed{\text{Equation viewpoint}}$

$$\boxed{\begin{aligned} 5x - 10y + 15z &= 3 \\ x + 3y + 5z &= 2 \\ x + 2z &= 0 \end{aligned}} \xrightarrow{\frac{1}{5}R_1 \to R_1} \boxed{\begin{aligned} x - 2y + 3z &= 3/5 \\ x + 3y + 5z &= 2 \\ x + 2z &= 0 \end{aligned}}$$

$\boxed{\text{Matrix viewpoint}}$

$$\left(\begin{array}{ccc|c} 5 & -10 & 15 & 3 \\ 1 & 3 & 5 & 2 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow{\frac{1}{5}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 3/5 \\ 1 & 3 & 5 & 2 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

Ex: (Interchanging two rows)

$\boxed{\text{Equation viewpoint}}$

$$\boxed{\begin{aligned} 5x - y &= 1 \\ x + 2y &= 3 \end{aligned}} \xrightarrow{R_1 \leftrightarrow R_2} \boxed{\begin{aligned} x + 2y &= 3 \\ 5x - y &= 1 \end{aligned}}$$

$\boxed{\text{Matrix viewpoint}}$

$$\left(\begin{array}{cc|c} 5 & -1 & 1 \\ 1 & 2 & 3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 5 & -1 & 1 \end{array}\right)$$

Ex: (Add a multiple of one row/eqn to another row/eqn)

Equation viewpoint

$$\begin{aligned} x + y - z &= 1 \\ 2x + y + 2z &= 0 \\ x - y - z &= 3 \end{aligned} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{aligned} x + y - z &= 1 \\ -y + 4z &= -2 \\ x - y - z &= 3 \end{aligned}$$

$$\begin{array}{rl} -2x - 2y + 2z = -2 & \leftarrow \boxed{-2R_1} \\ +\quad 2x + y + 2z = 0 & \leftarrow \boxed{R_2} \\ \hline -y + 4z = -2 & \leftarrow \boxed{\text{new } R_2} \end{array}$$

Matrix viewpoint

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & -1 & -1 & 3 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & 4 & -2 \\ 1 & -1 & -1 & 3 \end{array}\right)$$

$$
\begin{array}{r}
\begin{pmatrix} -2 & -2 & 2 & | & -2 \end{pmatrix} \leftarrow \boxed{-2R_1} \\
+ \begin{pmatrix} 2 & 1 & 2 & | & 0 \end{pmatrix} \leftarrow \boxed{R_2} \\
\hline
\begin{pmatrix} 0 & -1 & 4 & | & -2 \end{pmatrix} \leftarrow \boxed{\text{new } R_2}
\end{array}
$$