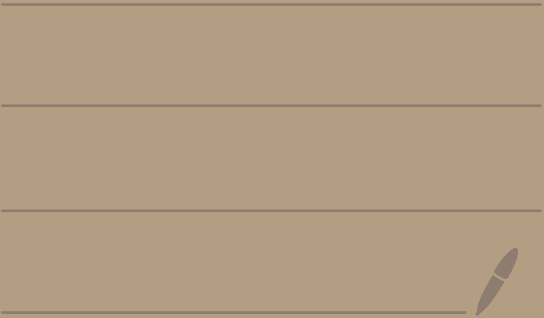


Math 2550-04  
9/30/24

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(Topic 4 continued...)

Last time we showed how to write a system of linear equations in the form  $A\vec{x} = \vec{b}$

If  $A^{-1}$  exists then we will be able to solve for  $\vec{x}$  as follows:

$$A\vec{x} = \vec{b}$$

$$\underbrace{A^{-1}A}_{I_n}\vec{x} = A^{-1}\vec{b}$$

$$I_n\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

So, if  $A^{-1}$  exists there will be one solution to the system.

Ex: Solve

$$\begin{array}{r} 3x \quad \quad + 3z = 9 \\ x + y + 2z = -4 \\ -2x + 3y \quad = 5 \end{array}$$

(\*)

Write it as  $A\vec{x} = \vec{b}$  like this:

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

Check: If you multiply the left side you get:

$$\begin{pmatrix} 3x + 3z \\ x + y + 2z \\ -2x + 3y \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

Same as system (\*)

Rewriting we have

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

Last week we found  $A^{-1}$  and it was

$$A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$$

Multiply both sides of  $\left[ \begin{matrix} \text{equation} \end{matrix} \right]$  by  $A^{-1}$  on the left to get

$$\underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

$$A^{-1}A = I_3$$

We get

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \cdot 9 + (-3)(-4) + (1)(5) \\ \frac{4}{3} \cdot 9 + (-2)(-4) + (1)(5) \\ -\frac{5}{3} \cdot 9 + (3)(-4) + (-1)(5) \end{pmatrix}$$

So,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 35 \\ 25 \\ -32 \end{pmatrix}$$

Thus, the answer to the system (\*) is

$$x = 35, y = 25, z = -32.$$

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## Topic 5 - Determinants

The determinant will allow us to detect if an  $n \times n$  matrix has an inverse.

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Def: Let  $A$  be an  $n \times n$  matrix. The matrix  $A_{ij}$  is defined to be the  $(n-1) \times (n-1)$  matrix obtained by deleting row  $i$  and column  $j$  from  $A$ .

---

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

remove row 2  
& column 3

$$A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

---

Def: Let  $A$  be an  $n \times n$  matrix. Let  $a_{ij}$  be the entry in row  $i$  and column  $j$ . The determinant of  $A$ , denoted by  $\det(A)$ ,

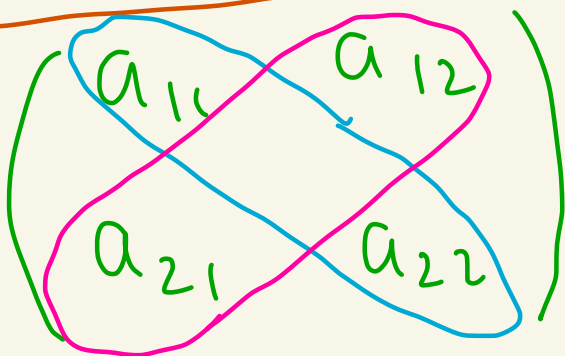
is defined as follows:

① If  $n=1$  and  $A = (a_{11})$

then  $\det(A) = a_{11}$

② If  $n=2$  and  $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

then  $\det(A) = a_{11}a_{22} - a_{12}a_{21}$



③ If  $n \geq 3$  then pick any column  $j$  to "expand on" and define

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

sums over the rows  $i$



column  $j$  is fixed

---

Note: In step 3 above you can instead pick a row  $i$  to expand on and then

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

$j=1$

sums over columns  $j$   
row  $i$  is fixed

---

Note: It doesn't matter what row or column you pick in step 3. You'll get the same answer in the end

---

Notation: Another notation for determinants is bars like this

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Ex:  $\det(3) = 3$

---

Ex:  $\det \begin{pmatrix} 1 & 5 \\ -2 & 6 \end{pmatrix}$

$$= (1)(6) - (5)(-2)$$

$$= 6 + 10$$

$$= 16$$

---

Ex:

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

Calculate  $\det(A)$

Let's pick column  $j = 3$  to expand on.

$$\det(A) = \sum_{\bar{i}=1}^3 (-1)^{\bar{i}+3} \cdot a_{\bar{i}3} \cdot \det(A_{\bar{i}3})$$

$$= (-1)^{1+3} \cdot a_{13} \cdot \det(A_{13}) \leftarrow \bar{i}=1$$

$$+ (-1)^{2+3} \cdot a_{23} \cdot \det(A_{23}) \leftarrow \bar{i}=2$$

$$+ (-1)^{3+3} \cdot a_{33} \cdot \det(A_{33}) \leftarrow \bar{i}=3$$

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$= (1)(0) \cdot \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$\leftarrow \begin{pmatrix} \cancel{3} & 1 & \cancel{0} \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (-1)(3) \cdot \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix}$$

$$\leftarrow \begin{pmatrix} 3 & 1 & 0 \\ \cancel{-2} & -4 & \cancel{3} \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (1)(-2) \cdot \begin{vmatrix} 3 & 1 \\ -2 & -4 \end{vmatrix}$$

$$\leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ \cancel{5} & 4 & \cancel{-2} \end{pmatrix}$$

$$= 0$$

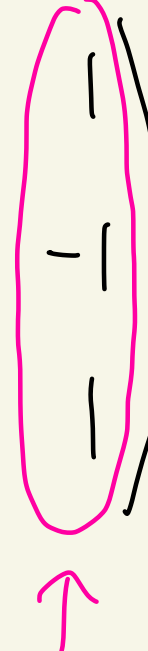
$$- 3 \cdot [(3)(4) - (1)(5)]$$

$$- 2 \cdot [(3)(-4) - (1)(-2)]$$

$$= -3[7] - 2[-10] = -1$$

# PICTURE WAY TO FIND $(-1)^{\bar{i}+\bar{j}}$

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$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$


We expanded on column 3