Math 2550-04
9/30/24

(Topic 4 continued...)

Last time we showed how to write a system of linear equations in the form $A\vec{x} = \vec{b}$

If A^{-1} exists then we will be able to solve for $\vec{x}$ as follows:

$$A\vec{x} = \vec{b}$$

$$\underbrace{A^{-1}A}_{I_n}\vec{x} = A^{-1}\vec{b}$$

$$I_n\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

So, if A^{-1} exists there will be one solution to the system.

Ex: Solve

$$\boxed{\begin{aligned} 3x \phantom{{}+y} &+ 3z = 9 \\ x + y &+ 2z = -4 \\ -2x + 3y \phantom{{}+2z} &= 5 \end{aligned}} \quad (*)$$

Write it as $A\vec{x} = \vec{b}$ like this:

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

Check: If you multiply the left side you get:

$$\begin{pmatrix} 3x + 3z \\ x + y + 2z \\ -2x + 3y \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \leftarrow \boxed{\text{same as system } (*)}$$

Rewriting we have

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

Last week we found A^{-1} and it was

$$A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$$

multiply both sides of by A^{-1} on the left to get

$$\underbrace{\underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_{A}}_{A^{-1}A = I_3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}$$

We get

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2\cdot 9 + (-3)(-4) + (1)(5) \\ \frac{4}{3}\cdot 9 + (-2)(-4) + (1)(5) \\ -\frac{5}{3}\cdot 9 + (3)(-4) + (-1)(5) \end{pmatrix}$$

So,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 35 \\ 25 \\ -32 \end{pmatrix}$$

Thus, the answer to the system $(*)$ is

$x = 35,\ y = 25,\ z = -32.$

Topic 5 – Determinants

The determinant will allow us to detect if an $n \times n$ matrix has an inverse.

Def: Let A be an $n \times n$ matrix. The matrix A_{ij} is defined to be the $(n-1)\times(n-1)$ matrix obtained by deleting row i and column j from A.

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 2 & \cancel{3} \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & 8 & \cancel{9} \end{pmatrix}$$

remove row 2 & column 3

$$A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} \leftarrow \begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{pmatrix}$$

<u>Def:</u> Let A be an $n \times n$ matrix. Let a_{ij} be the entry in row i and column j. The <u>determinant</u> of A, denoted by $\det(A)$,

is defined as follows:

① If $n=1$ and $A = (a_{11})$ then $\det(A) = a_{11}$

② If $n=2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

then $\underbrace{\det(A) = a_{11}a_{22} - a_{12}a_{21}}_{}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

③ If $n \geq 3$ then pick any column j to "expand on" and define

$$\det(A) = \underbrace{\sum_{i=1}^{n}}_{\text{sums over the rows } i} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

column j is fixed

Note: In step 3 above you can instead pick a row i to expand on and then

$$\det(A) = \underbrace{\sum_{j=1}^{n}}_{\text{sums over columns } j} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

row i is fixed

Note: It doesn't matter what row or column you pick in step 3. You'll get the same answer in the end

Notation: Another notation for determinants is bars like this

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Ex: $\det(3) = 3$

Ex: $\det\begin{pmatrix} 1 & 5 \\ -2 & 6 \end{pmatrix}$

$$= (1)(6) - (5)(-2)$$

$$= 6 + 10$$

$$= 16$$

Ex:

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

Calculate $\det(A)$

Let's pick column $j = 3$ to expand on.

$$\det(A) = \sum_{i=1}^{3} (-1)^{i+3} \cdot a_{i3} \cdot \det(A_{i3})$$

$$= (-1)^{1+3} \cdot a_{13} \cdot \det(A_{13}) \leftarrow i=1$$

$$+ (-1)^{2+3} \cdot a_{23} \cdot \det(A_{23}) \leftarrow i=2$$

$$+ (-1)^{3+3} \cdot a_{33} \cdot \det(A_{33}) \leftarrow i=3$$

$$\boxed{A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}}$$

$$= (1)(0)\cdot\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (-1)(3)\cdot\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (1)(-2)\cdot\begin{vmatrix} 3 & 1 \\ -2 & -4 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$= 0$$

$$- 3\cdot\left[(3)(4) - (1)(5)\right]$$

$$- 2\cdot\left[(3)(-4) - (1)(-2)\right]$$

$$= -3[7] - 2[-10] = -1$$

PICTURE WAY TO FIND $(-1)^{i+j}$

$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

We expanded on column 3