Math 2550-04

9/9/24

I'm redoing Topic 6 and
after on the website.
Both notes and HW
I'm keeping the old way I
did these later topics
at the bottom of the
website, but we won't use them.

Ex: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Recall $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Then,

$$I_2 A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{2\times 2}$$

✓

answer is 2×2

$$= \begin{pmatrix} (1\ 0)\cdot\begin{pmatrix} 1 \\ 3 \end{pmatrix} & (1\ 0)\cdot\begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ (0\ 1)\cdot\begin{pmatrix} 1 \\ 3 \end{pmatrix} & (0\ 1)\cdot\begin{pmatrix} 2 \\ 4 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 1 + 0\cdot 3 & 1\cdot 2 + 0\cdot 4 \\ 0\cdot 1 + 1\cdot 3 & 0\cdot 2 + 1\cdot 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = A$$

Also

$$AI_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 1 + 2\cdot 0 & 1\cdot 0 + 2\cdot 1 \\ 3\cdot 1 + 4\cdot 0 & 3\cdot 0 + 4\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = A$$

So, $AI_2 = A$ and $I_2 A = A$.

Ex: Let

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \quad \longleftarrow \quad \boxed{\begin{array}{c} 3\times 2 \\ \text{matrix} \end{array}}$$

Recall

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then,

$$I_3 B = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{3\times 3} \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}_{3\times 2}$$

answer is 3×2

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = B$$

So, $I_3 B = B$.

Can we do BI_3 ?

$$BI_3 = \underbrace{\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}}_{3\times 2}\underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}}_{3\times 3}$$

not the same

You can't do this multiplication

It's undefined.

But you could do this:

$$BI_2 = \underbrace{\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}}_{3\times 2}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{2\times 2} = \begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix} = B$$

same ✓

if you did the multiplication you'd get this

So, $BI_2 = B$.

Algebraic properties of matrices

Let A, B, C be matrices.
Let α, β be real numbers.
Then the following are true
where we will assume that the sizes of the matrices are such that the operations are defined:

① $A + B = B + A$

② $A + (B + C) = (A + B) + C$

③ $A(BC) = (AB)C$

④ $A(B + C) = AB + AC$

⑤ $(B + C)A = BA + CA$

⑥ $A(B-C) = AB - AC$

⑦ $(B-C)A = BA - CA$

⑧ $\alpha(B+C) = \alpha B + \alpha C$

⑨ $\alpha(B-C) = \alpha B - \alpha C$

⑩ $(\alpha+\beta)A = \alpha A + \beta A$

⑪ $(\alpha-\beta)A = \alpha A - \beta A$

⑫ $\alpha(\beta A) = (\alpha\beta)A$

⑬ $\alpha(AB) = (\alpha A)B = A(\alpha B)$

⑭ $(A^T)^T = A$

⑮ $(A+B)^T = A^T + B^T$

⑯ $(A-B)^T = A^T - B^T$

⑰ $(\alpha A)^T = \alpha A^T$

(18) $(AB)^T = B^T A^T$ ← note the order reverses

(19) If A is $m \times n$, then $AI_n = A$

(20) If A is $m \times n$, then $I_m A = A$.

(21) If A is $m \times n$, then $A - A = O_{m \times n}$

(22) If A is $m \times n$, then $A + O_{m \times n} = O_{m \times n} + A = A$

Let's prove 15 $(A+B)^T = A^T + B^T$ when A and B are both 2×2.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

The LHS gives:

$$(A+B)^T = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right)^T$$

$$= \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}^T$$

$$= \begin{pmatrix} a+e & c+g \\ b+f & d+h \end{pmatrix}$$

Also, the RHS gives:

$$A^T + B^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T + \begin{pmatrix} e & f \\ g & h \end{pmatrix}^T$$

EQUAL

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} + \begin{pmatrix} e & g \\ f & h \end{pmatrix}$$

$$= \begin{pmatrix} a+e & c+g \\ b+f & d+h \end{pmatrix}$$

Thus, $(A+B)^T = A^T + B^T$.

$\blacksquare$

Topic 3 – Systems of linear equations

<u>Def:</u> A <u>linear equation</u> in the n variables $x_1, x_2, \dots, x_n$ is an equation of the form

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b} \quad (*)$$

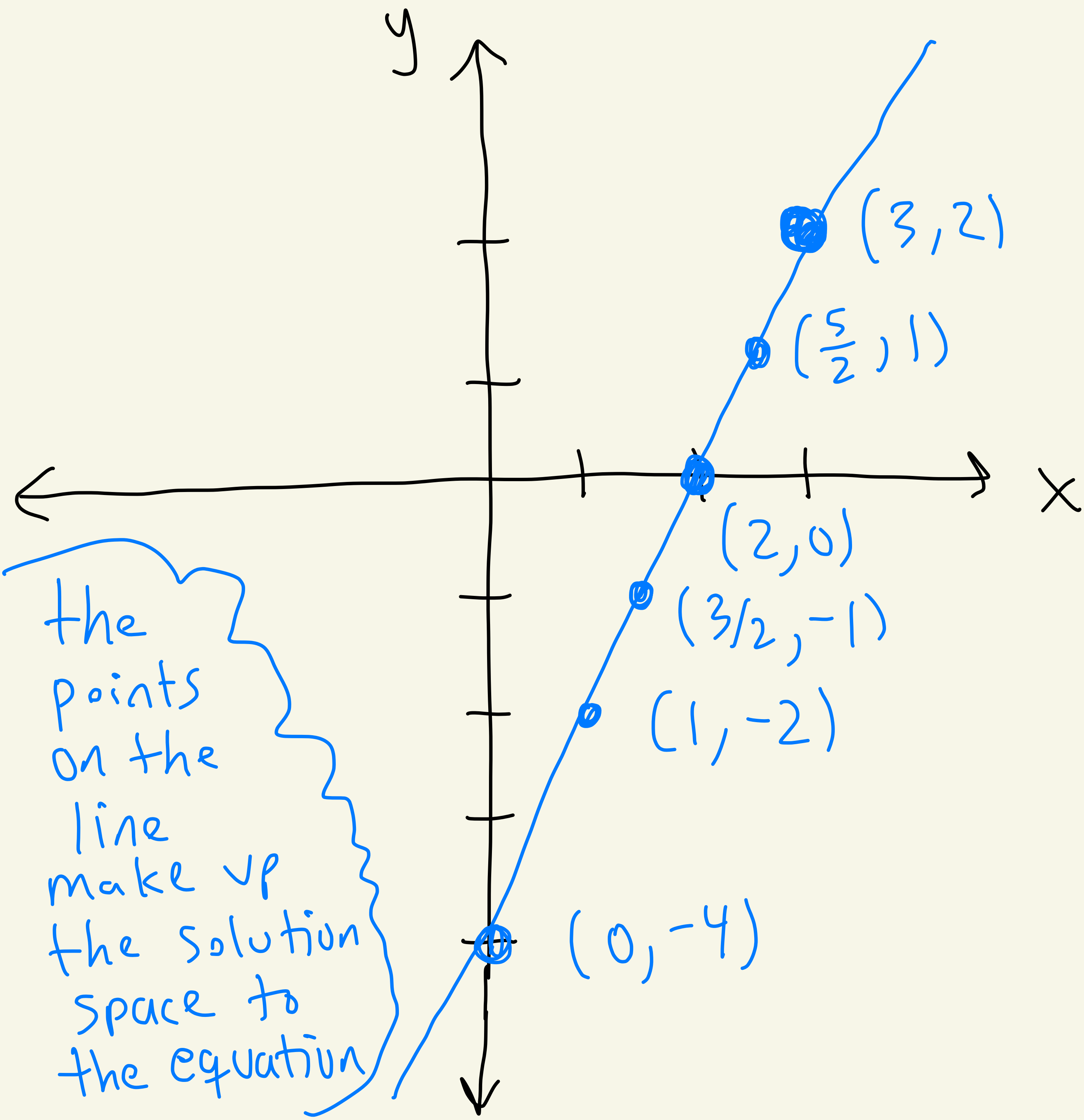
Where $a_1, a_2, \dots, a_n, b$ are constant real numbers.

The <u>solution space</u> of $(*)$ consists of the set of all $(x_1, x_2, \dots, x_n)$ that solve $(*)$.

Ex: Consider

$$4x - 2y = 8$$

linear eq. in 2 variables x, y



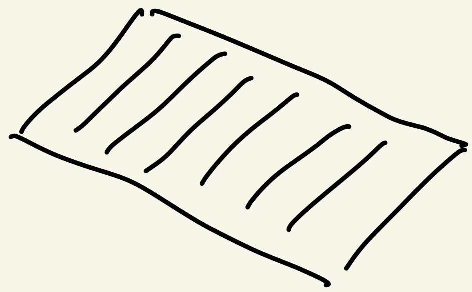
the points on the line make up the solution space to the equation

Ex: Consider the equation

$$2x - 3y + z = 5$$

linear equation in 3 variables

In Calculus, you learn this is a plane in 3d.



Some points in the solution space are:

$$(x,y,z) = (2,0,1),\ (4,1,0),\ (0,0,5),\ \ldots$$

Some linear equations:

$$10x - 3y + 5z - w + \frac{13}{2}t = 7$$

$$x_1 - x_2 + 7x_3 = 0$$

Some non-linear equations:

$$x - 13y^2 = 6$$

$$12x - \sqrt{y} = 10$$

$$x + 5\cos(y) = e^x$$