2550

HW 7 – Part 2

Solutions

① (a)

$W = \{ \langle a, 0, 0 \rangle \mid a \in \mathbb{R} \}$

$= \{ \langle 0,0,0 \rangle, \langle \pi, 0, 0 \rangle, \langle \frac{1}{2}, 0, 0 \rangle, \langle -1, 0, 0 \rangle, \ldots \}$

infinitely many more

Let $\langle a, 0, 0 \rangle$ be in W.

Then,

$\langle a, 0, 0 \rangle = a \cdot \langle 1, 0, 0 \rangle$

[Note that $\langle 1, 0, 0 \rangle$ is in W also.]

Thus, $\langle 1, 0, 0 \rangle$ spans W.

So, $W = \text{span}(\{ \langle 1, 0, 0 \rangle \})$.

Let $\beta = \{ \langle 1, 0, 0 \rangle \}$.

β is a linearly independent set since if

$$c_1 \langle 1, 0, 0 \rangle = \underbrace{\langle 0, 0, 0 \rangle}_{\vec{0}}$$

then

$$\langle c_1, 0, 0 \rangle = \langle 0, 0, 0 \rangle$$

and thus $c_1 = 0$.

Thus, $\beta = \{\langle 1,0,0\rangle\}$ is a linearly independent set that spans W and hence is a basis for W.

Thus, W is 1-dimensional.

That is $\dim(W) = 1$.

---

① (b) Let

$$W = \{\langle a,b,c\rangle \mid b = a+c \text{ where } a,b,c \in \mathbb{R}\}$$

$$= \{\langle 1,2,1\rangle, \langle 0,1,1\rangle, \langle \pi, \pi+2, 2\rangle, \ldots\}$$

(in W since $2=1+1$) (in W since $1=0+1$) (in W since $\pi+2=\pi+2$) (infinitely many more)

Suppose $\langle a,b,c\rangle$ is in W.

So, $b = a+c$.

Then

$$\langle a,b,c\rangle = \langle a, a+c, c\rangle = \langle a,a,0\rangle + \langle 0,c,c\rangle$$

$$= a\cdot\langle 1,1,0\rangle + c\cdot\langle 0,1,1\rangle$$

Note that $\langle 1,1,0\rangle$ and $\langle 0,1,1\rangle$ are in W.

Thus, $\langle 1,1,0\rangle$ and $\langle 0,1,1\rangle$ span W.

Let $\beta = \{\langle 1,1,0\rangle, \langle 0,1,1\rangle\}$

Then β spans W.

Let's show β is a linearly independent set.

Suppose

$$c_1\langle 1,1,0\rangle + c_2\langle 0,1,1\rangle = \underbrace{\langle 0,0,0\rangle}_{\vec{0}}$$

Then

$$\langle c_1, c_1, 0\rangle + \langle 0, c_2, c_2\rangle = \langle 0,0,0\rangle$$

So,

$$\langle c_1, c_1 + c_2, c_2\rangle = \langle 0,0,0\rangle$$

Thus,

$$\boxed{\begin{aligned} c_1 \quad\; &= 0 \\ c_1 + c_2 &= 0 \\ c_2 &= 0 \end{aligned}}$$

We see that the only solutions to these equations are $c_1 = 0$, $c_2 = 0$

Thus, $\beta = \{\langle 1,1,0\rangle, \langle 0,1,1\rangle\}$ forms ↓

a linearly independent set.

Since $\beta = \{\langle 1,1,0\rangle, \langle 0,1,1\rangle\}$ forms a linearly independent set and spans W, it is a basis for W.

Thus, W is 2-dimensional.

That is $\dim(W) = 2$.

② $V = M_{2,2}$, $F = \mathbb{R}$

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a+b+c=0,\ a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} 1 & -2 \\ 1 & 5 \end{pmatrix}}_{\substack{\text{in } W \\ \text{since} \\ 1+(-2)+1=0}}, \underbrace{\begin{pmatrix} 2 & 5 \\ -7 & \frac{1}{2} \end{pmatrix}}_{\substack{\text{in } W \\ \text{since} \\ 2+5+(-7)=0}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\substack{\text{in } W \\ \text{since} \\ 0+0+0=0}}, \dots \right\}$$

Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in W.

Then $a+b+c=0$.

Thus, $a=-b-c$.

So,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b-c & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} -b & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -c & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$= b\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that $\underbrace{\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}}_{\substack{\text{in } W \\ \text{since} \\ -1+1+0=0}}, \underbrace{\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}}_{\substack{\text{in } W \\ \text{since} \\ -1+0+0=0}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\text{in } W \\ \text{since} \\ 0+0+0=0}}$ are in W

Since every element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in W satisfies

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = b\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we have that

$$\beta = \left\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

spans W.

Let's show β is a linearly independent set.

Suppose

$$c_1\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\vec{0}}$$

Then

$$\begin{pmatrix} -c_1 & c_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -c_2 & 0 \\ c_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So,

$$\begin{pmatrix} -c_1 - c_2 & c_1 \\ c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus,

$$\boxed{\begin{aligned} -c_1 - c_2 &= 0 \\ c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}}$$

The only solutions to this system are $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Thus,

$$\beta = \left\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a linearly independent set.

Since β is a linearly independent set that spans W, we know that β is a basis for W.

Thus, W is 3-dimensional.

So, $\dim(W) = 3$.

③ In HW 6 you showed that

$$W = \left\{ a+bx+cx^2+dx^3 \;\middle|\; \begin{array}{l} a+b+c+d=0 \\ a,b,c,d \in \mathbb{R} \end{array} \right\}$$

is a subspace of $V = P_3$ over $F = \mathbb{R}$

Find a basis for W and state the dimension of W.

Examples of elements of W

$1 - x + x^2 - x^3$ because $1 + (-1) + (1) + (-1) = 0$

$1 + x + 2x^2 - 4x^3$ because $1 + 1 + 2 + (-4) = 0$

Need to solve

$$a+b+c+d=0$$

← system with leading variable is a; free variables are b, c, d

$b = t$
$c = s$
$d = u$
$a = -b-c-d = -t-s-u$

where t, s, u are any real #s

So, if $a+bx+cx^2+dx^3$ in W then $a+b+c+d=0$ and so

$$a+bx+cx^2+dx^3$$
$$= (-t-s-u)+tx+sx^2+ux^3$$
$$= [-t+tx]+[-s+sx^2]+[-u+ux^3]$$
$$= t[-1+x]+s[-1+x^2]+u[-1+x^3]$$

which is in $\text{span}(\{-1+x, -1+x^2, -1+x^3\})$

Note $-1+x, -1+x^2, -1+x^3$ are in W since their coefficients sum to 0.

So, $W = \text{span}(\{-1+x, -1+x^2, -1+x^3\})$

Let's check if

$$-1+x, \quad -1+x^2, \quad -1+x^3$$

are linearly independent.

Consider

$$c_1(-1+x) + c_2(-1+x^2) + c_3(-1+x^3) = \underbrace{0+0x+0x^2+0x^3}_{\vec{0} \text{ in } P_3}$$

What are the solutions?

We have that the above equation becomes

$$-c_1 + c_1 x - c_2 + c_2 x^2 - c_3 + c_3 x^3 = 0 + 0x + 0x^2 + 0x^3$$

Regrouping gives

$$(-c_1 - c_2 - c_3) + c_1 x + c_2 x^2 + c_3 x^3 = 0 + 0x + 0x^2 + 0x^3$$

Thus,

$$\begin{aligned} -c_1 - c_2 - c_3 &= 0 \quad (1)\\ c_1 \qquad\qquad &= 0 \quad (2)\\ c_2 \qquad &= 0 \quad (3)\\ c_3 &= 0 \quad (4) \end{aligned}$$

By (2), (3), (4) we get

$c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Since this is the only solution to

$c_1(-1+x) + c_2(-1+x^2) + c_3(-1+x^3) = 0 + 0x + 0x^2 + 0x^3$

the vectors $-1+x$, $-1+x^2$, $-1+x^3$ are linearly independent and thus form a basis for W.

basis for W	dimension of W
$-1+x, -1+x^2, -1+x^3$	3