Math 2550-01

10/16/24

Topic 6 - Coordinate systems in $\mathbb{R}^n$

beta

Def: Let $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a set of r vectors in $\mathbb{R}^n$.

- We say that a vector $\vec{v}$ is in the span of $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ if we can write

$$\vec{v} = \boxed{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r}$$

where $c_1, c_2, \ldots, c_r$ are real numbers.

(called a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$)

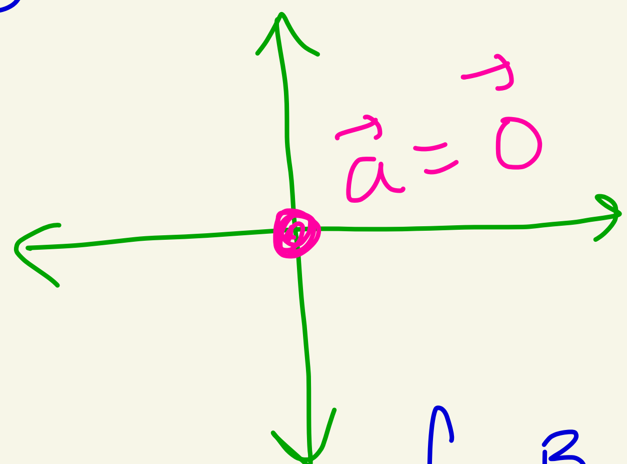
- Suppose $r = 1$ and $\beta = \{\vec{v}_1\}$.
  If $\vec{v}_1 = \vec{0}$, then we say β is a <u>linearly dependent</u> set
  If $\vec{v}_1 \neq \vec{0}$, then we say that β is a <u>linearly independent</u> set.

- Suppose $r \geq 2$, so β has 2 or more vectors. If one of the vectors in β can be written as a linear combination of the other vectors, then β is called <u>linearly dependent</u>

If not β is called <u>linearly independent</u>.

<u>Ex:</u> Let $\vec{a} = \langle 0, 0 \rangle$ in $\mathbb{R}^2$.

$\beta = \{\vec{a}\}$.



The vectors in the span of β are of the form

$$c_1\vec{a} = c_1\vec{0} = \vec{0}$$

for example

$$1 \cdot \vec{0} = \vec{0}$$
$$2 \cdot \vec{0} = \vec{0}$$

$-5 \cdot \vec{0} = \vec{0}$

You just get $\vec{0}$ every time.

Since $\beta = \{\vec{0}\}$ by def, β is called a linearly dependent set.

<u>Ex</u>: Let $\vec{v} = \langle 1, 2 \rangle$ in $\mathbb{R}^2$.

$\beta = \{\vec{v}\}$.

What are some vectors in the span of β?

They are of the form $c_1 \vec{v}$.

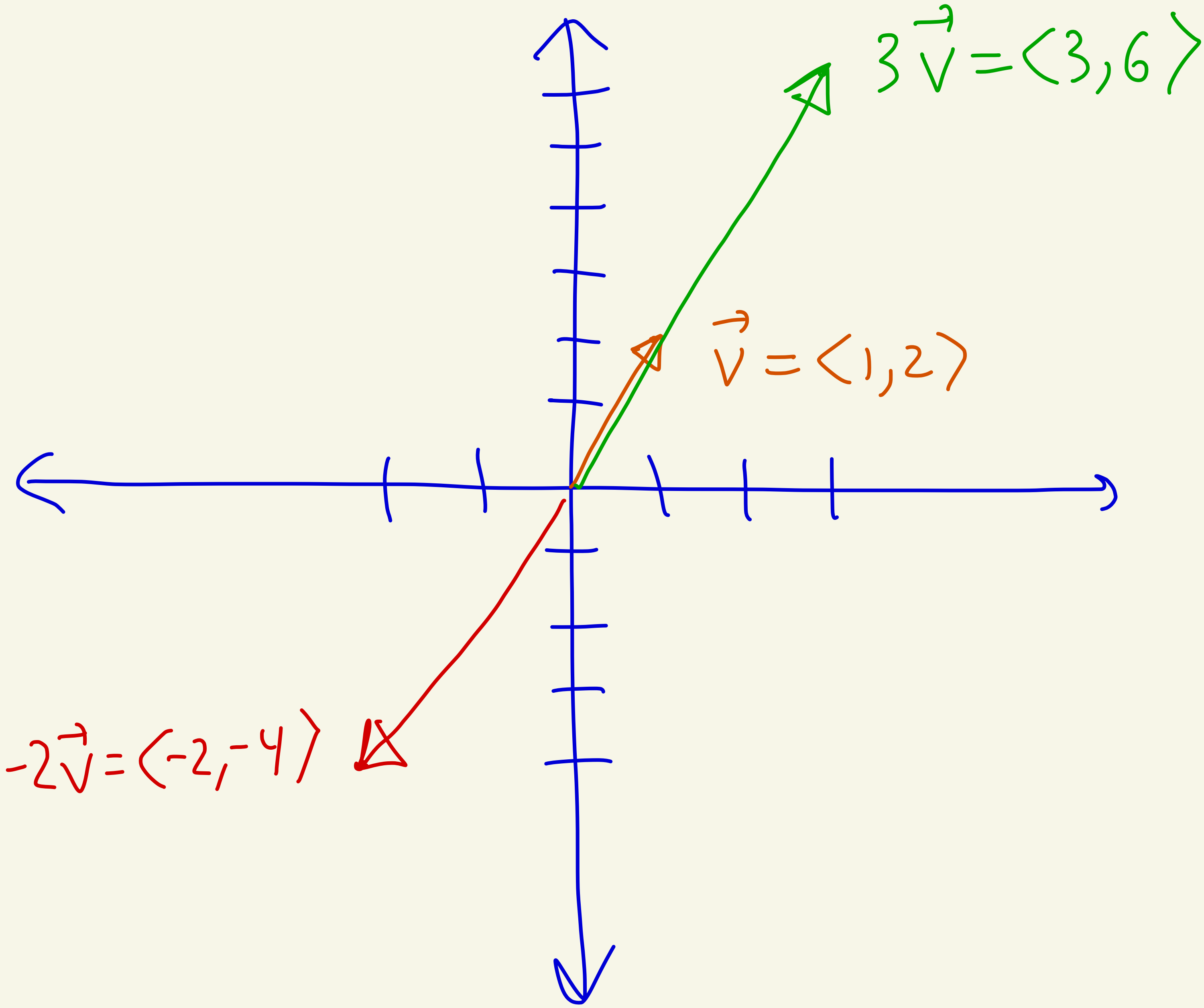
So some are:

$$3\vec{v} = 3\langle 1,2\rangle = \langle 3,6\rangle$$

$$1\cdot\vec{v} = \langle 1,2\rangle$$

$$-2\cdot\vec{v} = -2\langle 1,2\rangle = \langle -2,-4\rangle$$

and so on.



The span of $\vec{v}$ consists of all vectors on the line that $\vec{v}$ makes.

<u>Ex:</u> Let $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$ be in $\mathbb{R}^2$. Let $\beta = \{\vec{i}, \vec{j}\}$.

What are the vectors in the span of β? These vectors are of the form

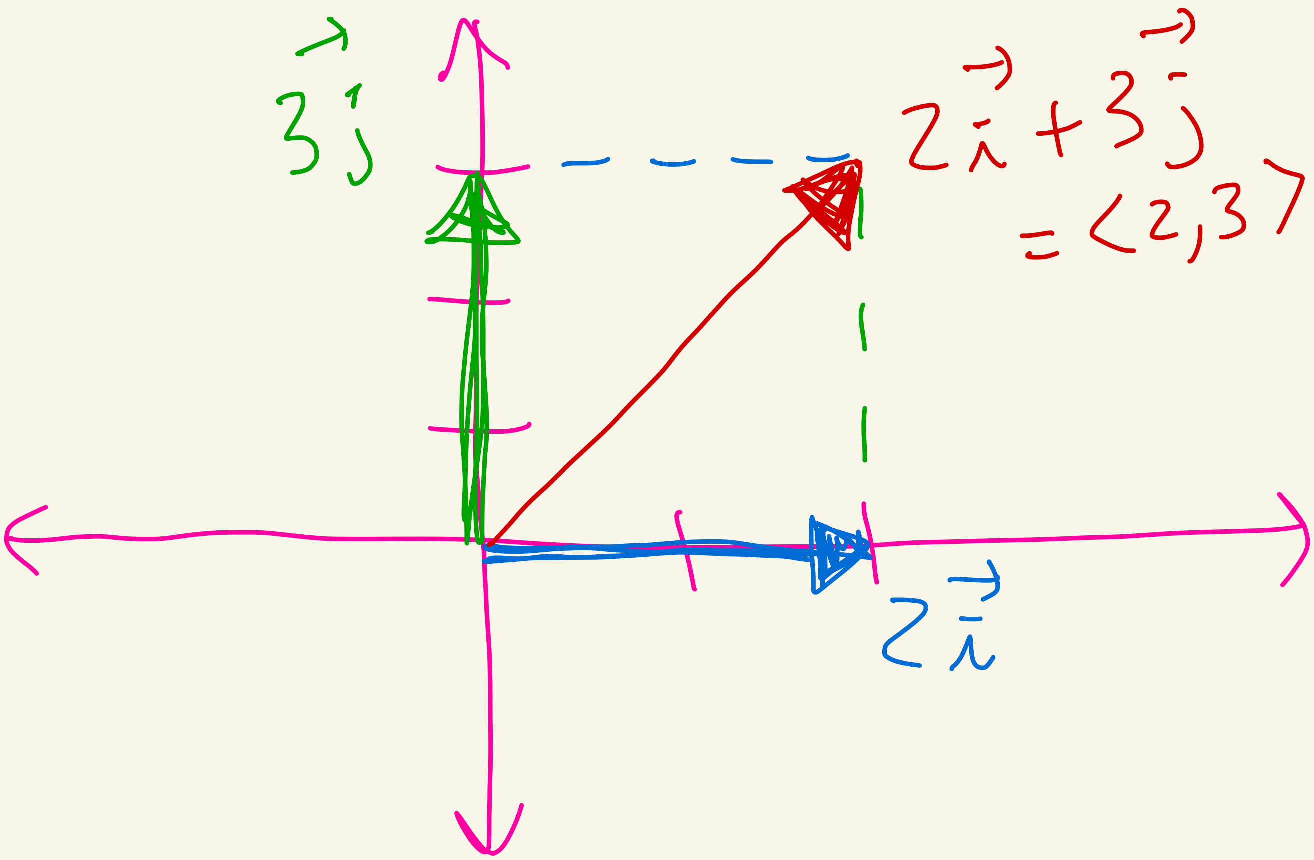
$$c_1 \vec{i} + c_2 \vec{j}$$

So some examples are

$$2\vec{i} + 3\vec{j} = 2\langle 1, 0 \rangle + 3\langle 0, 1 \rangle$$
$$= \langle 2, 3 \rangle$$

$$-1\cdot\vec{i} + 1\cdot\vec{j} = -\langle 1,0\rangle + \langle 0,1\rangle$$
$$= \langle -1, 1\rangle$$

So, $\langle 2,3\rangle$ and $\langle -1,1\rangle$ are in the span of $\beta = \{\vec{i}, \vec{j}\}$.



Every vector is in the span of $\beta = \{\vec{i}, \vec{j}\}$ because if $\langle a,b\rangle$ is a vector then

$$\langle a,b\rangle = \langle a,0\rangle + \langle 0,b\rangle$$

$= a\langle 1,0\rangle + b\langle 0,1\rangle$
$= a\vec{i} + b\vec{j}$

<u>Q:</u> Is $\beta = \{\vec{i}, \vec{j}\}$ a linearly dependent or linearly independent set? Is one of the vectors a linear combination of the other?

Can we make $\vec{i}$ from $\vec{j}$?
That is, is $\vec{i} = c_1 \vec{j}$?

Is $\underbrace{\langle 1,0\rangle}_{\vec{i}} = c_1 \underbrace{\langle 0,1\rangle}_{\vec{j}}$?

No because this would require
$\langle 1,0\rangle = \langle 0, c_1\rangle$

which would give $1=0$ which isn't possible.

Likewise $\vec{j}$ is not a multiple of $\vec{i}$ either. That is, you can't write $\underbrace{\langle 0,1\rangle}_{\vec{j}} = c_1 \underbrace{\langle 1,0\rangle}_{\vec{i}}$

The vectors are linearly independent.

<u>Ex:</u> Let $\vec{v}_1 = \langle 1,1\rangle$ and $\vec{v}_2 = \langle 2,2\rangle$ in $\mathbb{R}^2$.

Let $\beta = \{\vec{v}_1, \vec{v}_2\}$.

<u>Q:</u> Is β linearly dependent or linearly independent?

Note that

$$\vec{v}_2 = 2\vec{v}_1$$

So, $\vec{v}_2$ is a linear combination of $\vec{v}_1$.

Thus, $\beta = \{\vec{v}_1, \vec{v}_2\}$ is a linearly dependent set.

Note $\vec{v}_2 = 2\vec{v}_1$ can be written

$$2\vec{v}_1 - 1 \cdot \vec{v}_2 = \vec{0}$$

Q: What vectors lie in the span of $\beta = \{\vec{v}_1, \vec{v}_2\}$?

Some examples of vectors in the span are

$$1 \cdot \vec{v}_1 - 2\vec{v}_2 = 1 \cdot \langle 1,1 \rangle - 2 \cdot \langle 2,2 \rangle$$
$$= \boxed{\langle -3, -3 \rangle}$$

$$2 \cdot \vec{v}_1 + 2\vec{v}_2 = 2\langle 1,1 \rangle + 2\langle 2,2 \rangle$$
$$= \boxed{\langle 6, 6 \rangle}$$

Note any vector in the span of $\beta = \{\vec{v}_1, \vec{v}_2\}$ looks like

$$c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\langle 1,1 \rangle + c_2\langle 2,2 \rangle$$
$$= \langle c_1 + 2c_2, c_1 + 2c_2 \rangle$$
$$= (c_1 + 2c_2)\langle 1,1 \rangle$$
$$= (c_1 + 2c_2) \cdot \vec{v}_1$$

So, the span of $\beta = \{\vec{v}_1, \vec{v}_2\}$ is really just the span of $\vec{v}_1$ by itself.

Syllabus –

test 1 – 33.3%
test 2 – 33.3%
final – 33.3%

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drop 1 –

$$\max\{\text{test 1}, \text{test 2}\} = 50\%$$

$$\text{final} = 50\%$$

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no final –

test 1 – 50%
test 2 – 50%

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final

final – 100%