Math 2550-01

10/2/24

<u>Ex:</u> Last time we saw that

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} = -1$$

by expanding on column 3.

Let's calculate this again but expand on row 1.

$$\det\begin{pmatrix}3&1&0\\-2&-4&3\\5&4&-2\end{pmatrix}$$

$$\boxed{\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}}$$

$$=(1)(3)\begin{vmatrix}-4&3\\4&-2\end{vmatrix}+(-1)(1)\begin{vmatrix}-2&3\\5&-2\end{vmatrix}+(1)(0)\begin{vmatrix}-2&-4\\5&4\end{vmatrix}$$

$$\begin{pmatrix}3&1&0\\-2&-4&3\\5&4&-2\end{pmatrix}\qquad\begin{pmatrix}3&1&0\\-2&-4&3\\5&4&-2\end{pmatrix}\qquad\begin{pmatrix}3&1&0\\-2&-4&3\\5&4&-2\end{pmatrix}$$

$$\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}\qquad\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}\qquad\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}$$

$$=3\left[(-4)(-2)-(3)(4)\right]$$

$$-\left[(-2)(-2)-(5)(3)\right]$$

$$+0$$

$$= 3[-4] - [-11] = -1$$

Ex:

$$\det\begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

expand on column 4

$$= (-1)(-1)\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{vmatrix} \leftarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$+ (1)(0)\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{vmatrix} \leftarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

$$+(-1)(-2)\begin{vmatrix}1&0&0\\2&1&3\\1&1&2\end{vmatrix} \leftarrow \begin{pmatrix}1&0&0&-1\\2&1&3&0\\1&0&0&-2\\1&1&2&0\end{pmatrix}$$

$$+(1)(0)\begin{vmatrix}1&0&0\\2&1&3\\1&0&0\end{vmatrix} \leftarrow \begin{pmatrix}1&0&0&-1\\2&1&3&0\\1&0&0&-2\\1&1&2&0\end{pmatrix}$$

$$= \begin{vmatrix}2&1&3\\1&0&0\\1&1&2\end{vmatrix} + 2\begin{vmatrix}1&0&0\\2&1&3\\1&1&2\end{vmatrix}$$

$$= -1\cdot\begin{vmatrix}1&3\\1&2\end{vmatrix} + 0\cdot\begin{vmatrix}2&3\\1&2\end{vmatrix} - 0\cdot\begin{vmatrix}2&1\\1&1\end{vmatrix}$$

$$\begin{pmatrix}2&1&3\\1&0&0\\1&1&2\end{pmatrix} \quad \begin{pmatrix}2&1&3\\1&0&0\\1&1&2\end{pmatrix} \quad \begin{pmatrix}2&1&3\\1&0&0\\1&1&2\end{pmatrix}$$

$$\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}$$

$$+2\left[1\cdot\begin{vmatrix}1 & 3\\ 1 & 2\end{vmatrix} - 0 + 0\right]$$

$$\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 3\\ 1 & 1 & 2\end{pmatrix} \quad \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 3\\ 1 & 1 & 2\end{pmatrix} \quad \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 3\\ 1 & 1 & 2\end{pmatrix}$$

$$= -\begin{vmatrix}1 & 3\\ 1 & 2\end{vmatrix} + 2\begin{vmatrix}1 & 3\\ 1 & 2\end{vmatrix}$$

$$= \begin{vmatrix}1 & 3\\ 1 & 2\end{vmatrix} = \left[(1)(2) - (3)(1)\right]$$

$$= -1$$

Properties of the determinant

Let A and B be $n \times n$ matrices. Then:

1. $\det(A^T) = \det(A)$
2. $\det(AB) = \det(A) \cdot \det(B)$
3. If $\det(A) = 0$, then A^{-1} does not exist.
4. If $\det(A) \neq 0$, then A^{-1} does exist.
5. If A^{-1} exists, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$\boxed{\text{Formula for } A^{-1} \text{ if } A \text{ is } 2\times 2}$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $\det(A) = ad - bc$

If $ad - bc \neq 0$, then A^{-1} exists and

$$A^{-1} = \underbrace{\frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{\frac{1}{\det(A)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Ex: $A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$

$\det(A) = 4 - (-6) = 10 \neq 0.$

So, A^{-1} exists and

$$A^{-1} = \frac{1}{10} \cdot \begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4/10 & -3/10 \\ 2/10 & 1/10 \end{pmatrix}$$

$$= \begin{pmatrix} 2/5 & -3/10 \\ 1/5 & 1/10 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$

$\det(A) = (1)(-2) - (1)(-2) = 0$

A^{-1} does <u>not</u> exist.

<u>Note:</u> In general, if A is $n \times n$ and $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \cdot M$$

where M is the "adjugate matrix"