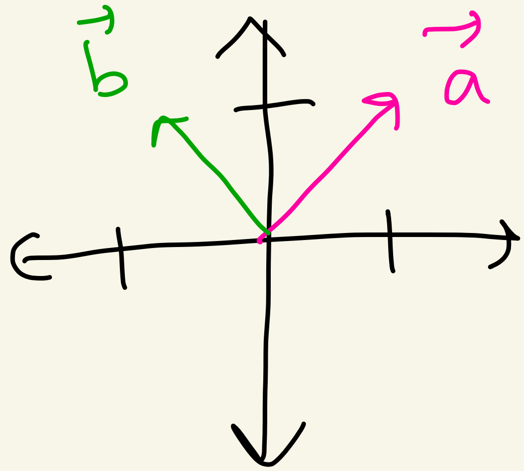
Math 2550-01
10/23/24

Last time we showed that $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$ are linearly independent.



Since $\beta = [\vec{a}, \vec{b}]$ consists of 2 linearly independent vectors in $\mathbb{R}^2$ we know that β is a basis / coordinate system for $\mathbb{R}^2$. So any vector $\vec{v}$ in $\mathbb{R}^2$ will be able to be written $\vec{v} = c_1 \vec{a} + c_2 \vec{b}$

where c_1, c_2 are unique numbers associated with $\vec{v}$, called its coordinates.

Let $\vec{v} = \langle 3, 1 \rangle$.

Let's find $\vec{v}$'s coordinates.

Need to solve

$$\underbrace{\langle 3, 1 \rangle}_{\vec{v}} = \underbrace{c_1 \langle 1, 1 \rangle + c_2 \langle -1, 1 \rangle}_{c_1 \vec{a} + c_2 \vec{b}}$$

This gives

$$\langle 3, 1 \rangle = \langle c_1, c_1 \rangle + \langle -c_2, c_2 \rangle$$

So,

$$\langle 3, 1 \rangle = \langle c_1 - c_2, c_1 + c_2 \rangle$$

Need to solve

$$\boxed{\begin{aligned} c_1 - c_2 &= 3 \quad (1) \\ c_1 + c_2 &= 1 \quad (2) \end{aligned}}$$

① + ② gives $2c_1 = 4$. So $\boxed{c_1 = 2}$

Plug into ② to get $2 + c_2 = 1$.

So, $\boxed{c_2 = -1.}$

Thus,

$$\underbrace{\langle 3, 1 \rangle}_{\vec{v}} = \underbrace{2\langle 1, 1 \rangle - 1 \cdot \langle -1, 1 \rangle}_{2\vec{a} - 1 \cdot \vec{b}}$$

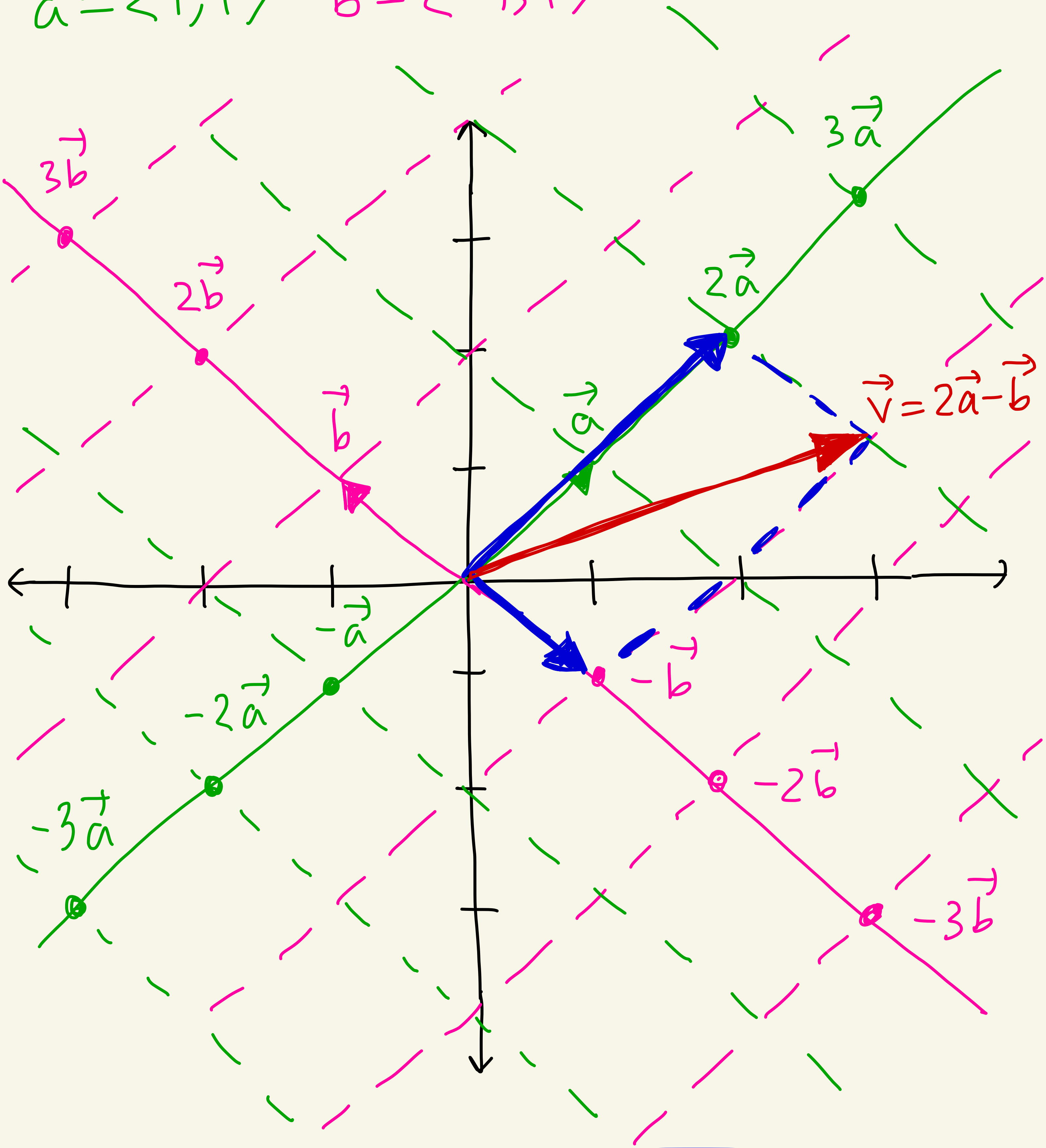
So,

$$\underbrace{[\vec{v}]_\beta}_{\vec{v}\text{'s } \beta\text{-coordinates}} = \langle 2, -1 \rangle$$

$$\boxed{\beta = [\vec{a}, \vec{b}]}$$

Let's draw a picture.

$\vec{a} = \langle 1, 1 \rangle \quad \vec{b} = \langle -1, 1 \rangle$



$$\boxed{\vec{v} = \langle 3, 1 \rangle = 2\vec{a} - 1 \cdot \vec{b}}$$

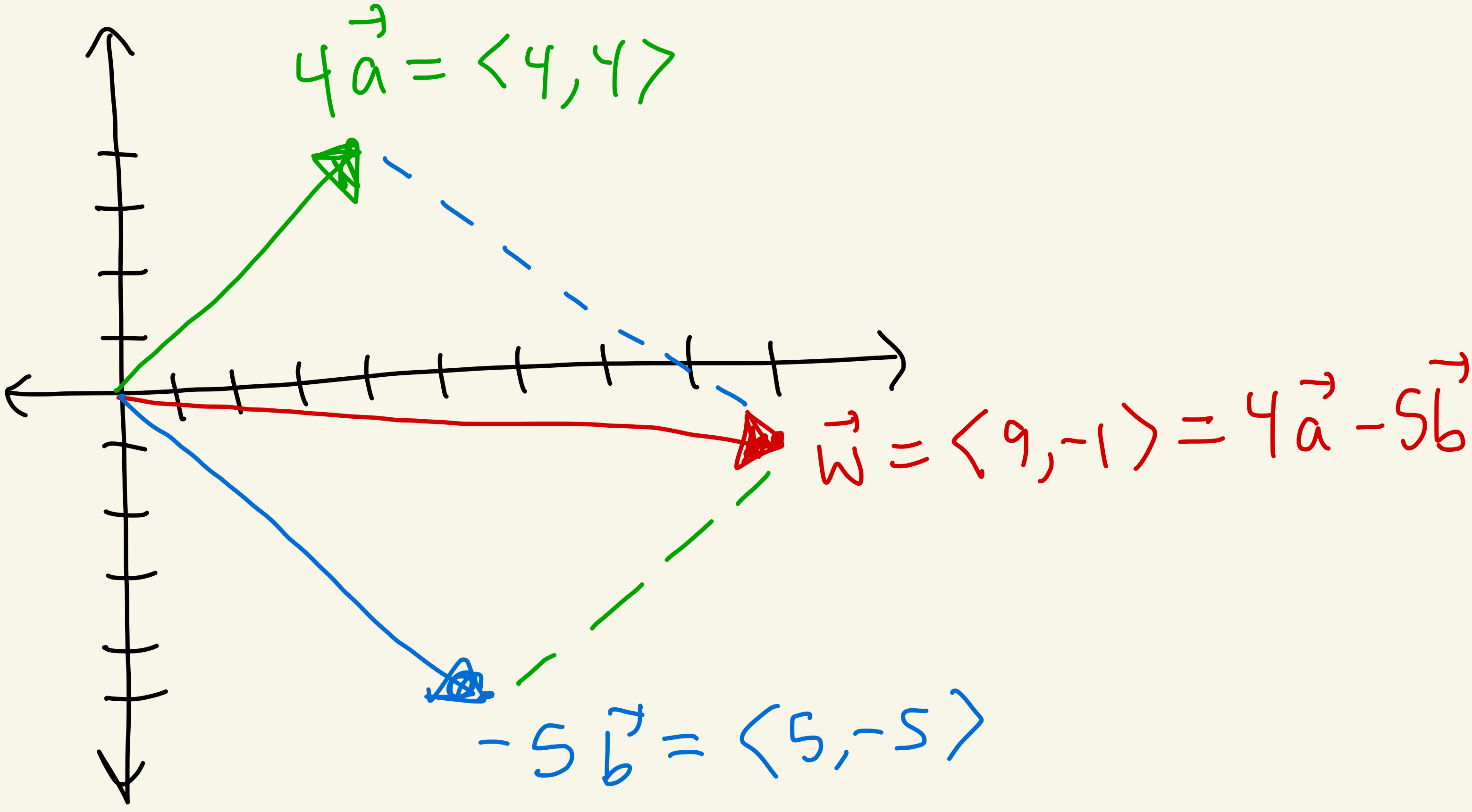
Suppose you know that $[\vec{w}]_\beta = \langle 4,-5 \rangle$. What is $\vec{w}$?

$\underbrace{[\vec{w}]_\beta}_{\vec{w}\text{'s } \beta\text{-coordinates}}$

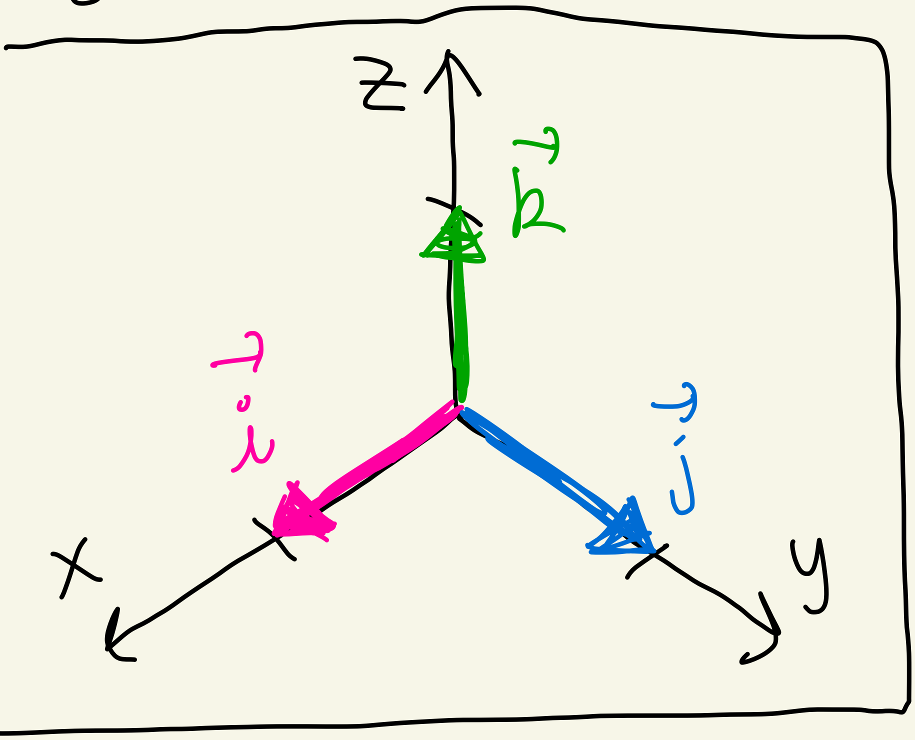
$\beta = [\vec{a}, \vec{b}]$, $\vec{a} = \langle 1,1 \rangle$, $\vec{b} = \langle -1,1 \rangle$

We get

$$\vec{w} = 4\vec{a} - 5\vec{b} = 4\langle 1,1 \rangle - 5\langle -1,1 \rangle$$
$$= \langle 9,-1 \rangle$$



Ex: In $\mathbb{R}^3$, let $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, $\vec{k} = \langle 0,0,1 \rangle$.



In the HW you will show that $\vec{i}, \vec{j}, \vec{k}$ are linearly independent.

So, $\beta = [\vec{i}, \vec{j}, \vec{k}]$ is a basis or coordinate system for $\mathbb{R}^3$.

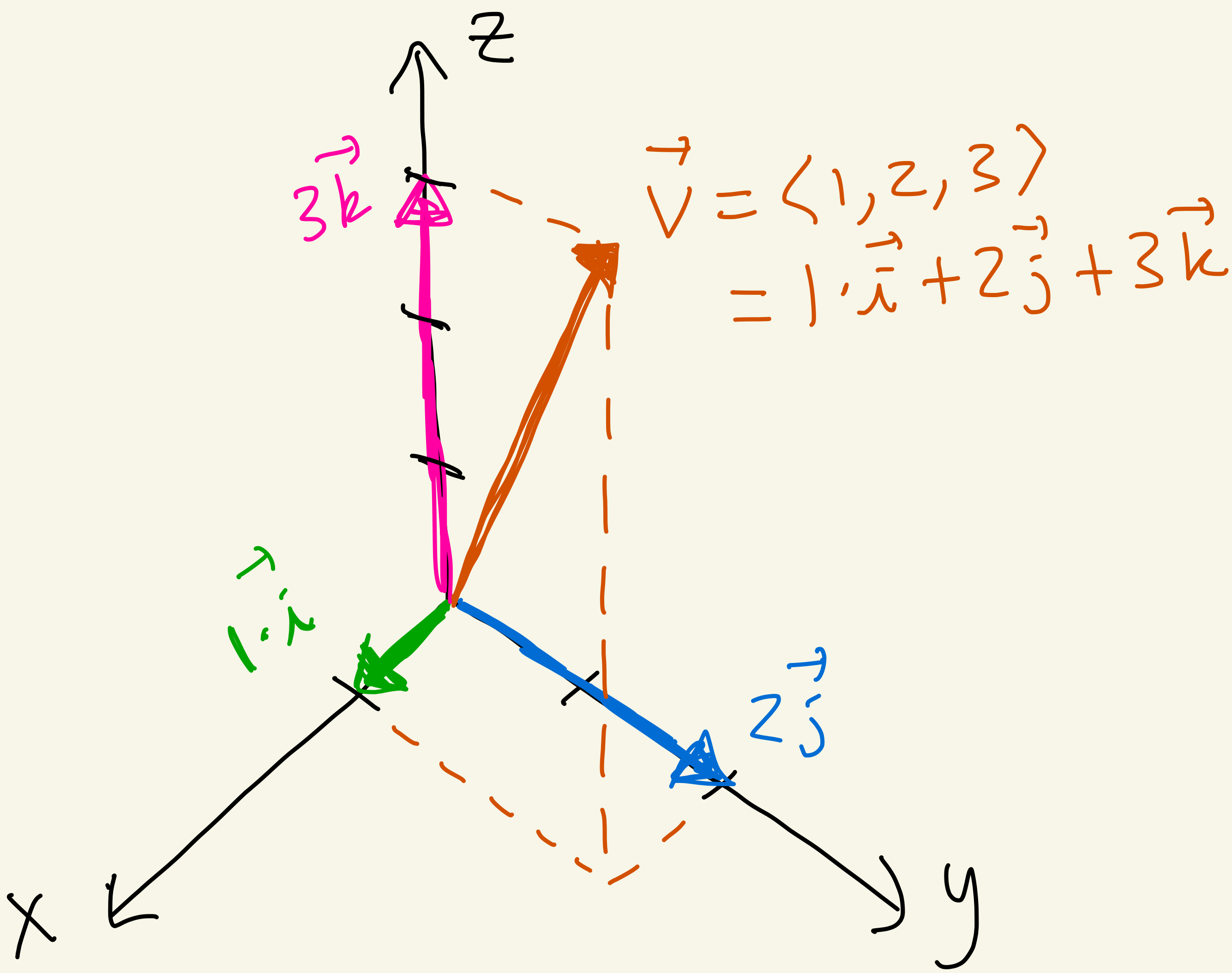
It's called the standard basis for $\mathbb{R}^3$.

If for example, $\vec{v} = \langle 1,2,3 \rangle$.

Then,

$$\vec{v} = \langle 1, 2, 3 \rangle$$
$$= \langle 1, 0, 0 \rangle + \langle 0, 2, 0 \rangle + \langle 0, 0, 3 \rangle$$
$$= 1 \cdot \langle 1, 0, 0 \rangle + 2 \langle 0, 1, 0 \rangle + 3 \langle 0, 0, 1 \rangle$$
$$= 1 \cdot \vec{i} + 2 \cdot \vec{j} + 3 \cdot \vec{k}$$

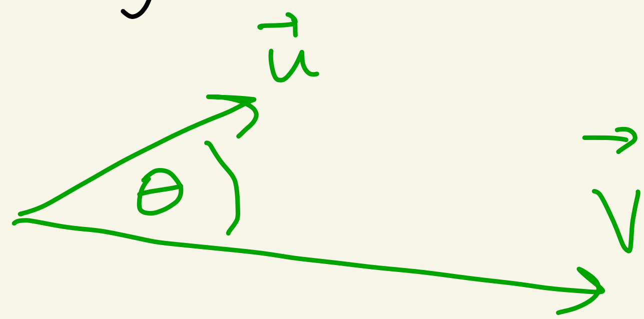
So,
$$[\vec{v}]_\beta = \langle 1, 2, 3 \rangle$$



Recall that if $\vec{u}$ and $\vec{v}$ are in $\mathbb{R}^2$ or $\mathbb{R}^3$ we have

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(\theta)$$

where θ is the angle between $\vec{u}$ and $\vec{v}$



So $\theta = 90^\circ$ precisely when

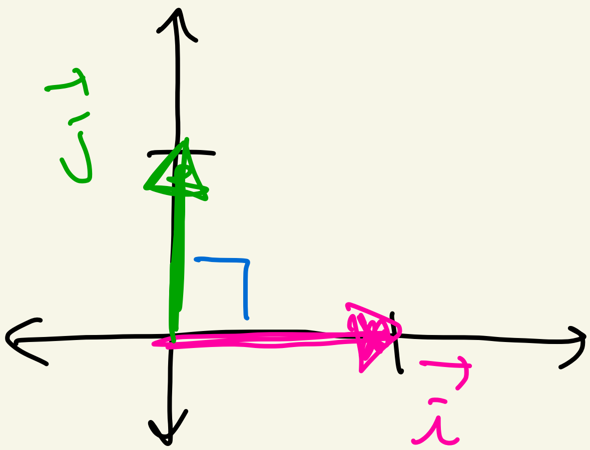
$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \underbrace{\cos(90^\circ)}_{0} = 0$$

<u>Def:</u> Given any two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$, we say that $\vec{a}$ and $\vec{b}$ are <u>orthogonal</u> if $\vec{a} \cdot \vec{b} = 0$.

or perpendicular

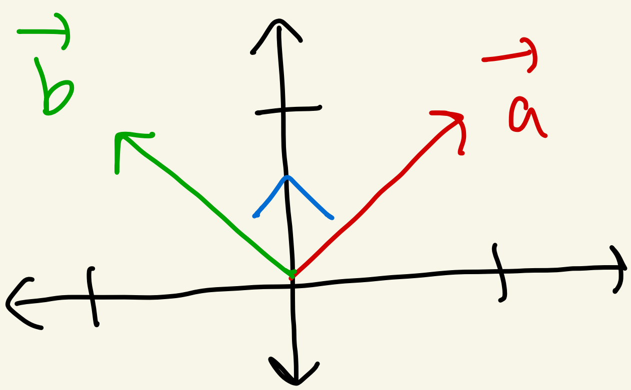
Ex: In $\mathbb{R}^2$, $\vec{i} = \langle 1,0 \rangle$, $\vec{j} = \langle 0,1 \rangle$

$\vec{i} \cdot \vec{j} = \langle 1,0 \rangle \cdot \langle 0,1 \rangle = (1)(0) + (0)(1) = 0$



So, $\vec{i}$ and $\vec{j}$ are orthogonal

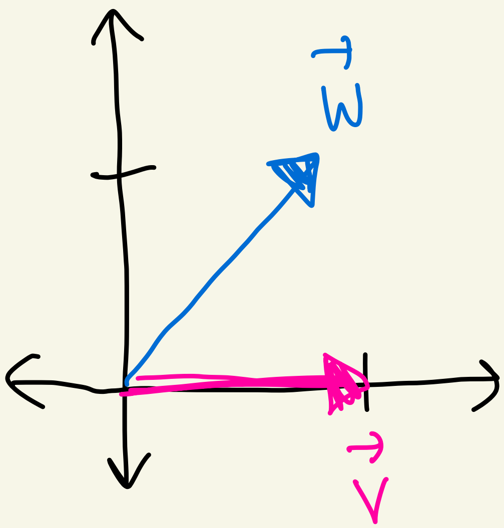
Ex: In $\mathbb{R}^2$, let $\vec{a} = \langle 1,1 \rangle$, $\vec{b} = \langle -1,1 \rangle$



$$\vec{a} \cdot \vec{b} = \langle 1,1 \rangle \cdot \langle -1,1 \rangle$$
$$= (1)(-1) + (1)(1)$$
$$= 0$$

So, $\vec{a}$ and $\vec{b}$ are orthogonal

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 1, 0 \rangle$, $\vec{w} = \langle 1, 1 \rangle$



Then,

$$\vec{v} \cdot \vec{w} = \langle 1, 0 \rangle \cdot \langle 1, 1 \rangle$$
$$= (1)(1) + (0)(1)$$
$$= 1 \neq 0$$

So, $\vec{v}$ and $\vec{w}$ are not orthogonal