Math 2550-01
10/28/24

(topic 6 continued...)

Def: Let $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ be a basis/coordinate system for $\mathbb{R}^n$

- We say that $\beta$ is an orthogonal basis if every pair of vectors in $\beta$ are orthogonal, that is $\vec{v}_a \cdot \vec{v}_b = 0$ if $a \neq b$.
- We say that $\beta$ is an orthonormal basis if

  (i) $\beta$ is orthogonal,

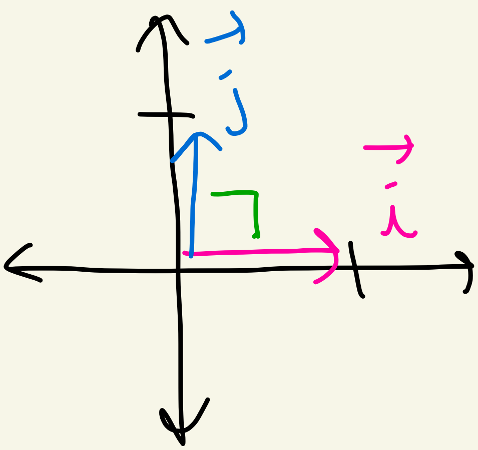
  and (ii) every vector in $\beta$ has length $\underline{1}$.

Ex: In $\mathbb{R}^2$, let $\beta = [\vec{i}, \vec{j}]$

where $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$

We saw before that $\beta$ is a basis.

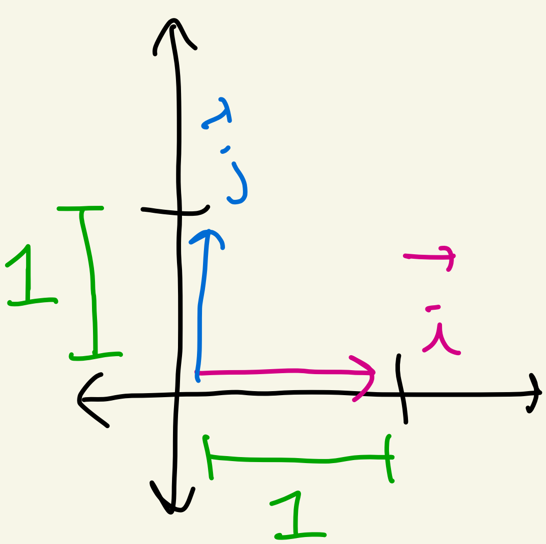
Q: Is $\beta$ an orthogonal basis?




$$\vec{i} \cdot \vec{j} = (1)(0) + (0)(1)$$
$$= 0$$

Yes, $\beta$ is an orthogonal basis

Q: Is $\beta$ an orthonormal basis?




(i) $\beta$ is orthogonal

(ii) $\|\vec{i}\| = \sqrt{1^2 + 0^2} = 1$

$\|\vec{j}\| = \sqrt{0^2 + 1^2} = 1$

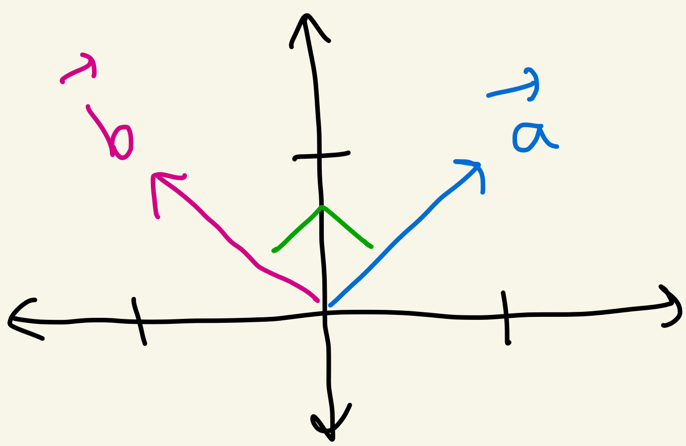
Yes, $\beta$ is an orthonormal basis.

**Ex:** In $\mathbb{R}^2$, let $\beta = [\vec{a}, \vec{b}]$

where $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$.

We saw previously that $\beta$ is a basis.

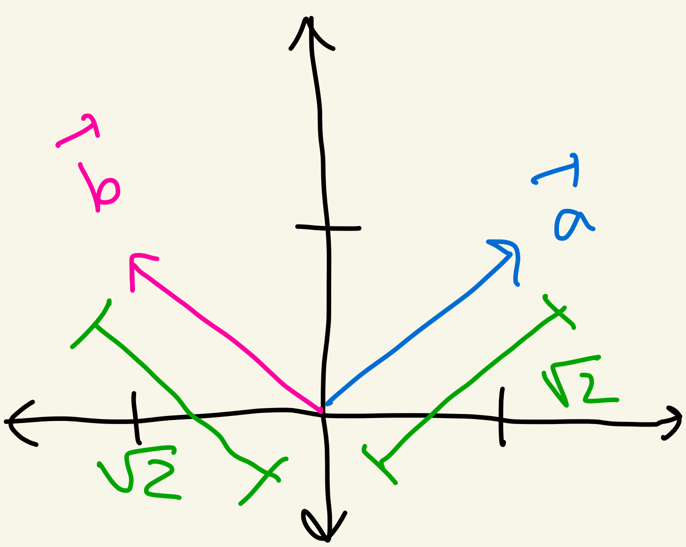
**Q:** Is $\beta$ an orthogonal basis?




$$\vec{a} \cdot \vec{b} = (1)(-1) + (1)(1)$$
$$= 0$$

Yes, $\beta$ is an orthogonal basis.

**Q:** Is $\beta$ an orthonormal basis?




(i) $\beta$ is orthogonal

(ii)

$$\|\vec{a}\| = \sqrt{1^2 + 1^2} = \sqrt{2} \neq 1$$
$$\|\vec{b}\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \neq 1$$

$\beta$ is not an orthonormal basis

Ex: (HW 6 #3)

In $\mathbb{R}^2$, let $\vec{a} = \langle 1,1 \rangle$, $\vec{b} = \langle 1,0 \rangle$.

- Let's show $\beta = [\vec{a}, \vec{b}]$ is a basis by showing that $\vec{a}$ and $\vec{b}$ are linearly independent.

Need to solve

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

for $c_1, c_2$.

We get

$$c_1 \langle 1,1 \rangle + c_2 \langle 1,0 \rangle = \langle 0,0 \rangle$$

which gives

$$\langle c_1, c_1 \rangle + \langle c_2, 0 \rangle = \langle 0,0 \rangle$$

$$\langle c_1 + c_2, c_1 \rangle = \langle 0, 0 \rangle$$

This gives

$$\boxed{\begin{aligned} c_1 + c_2 &= 0 \quad (1) \\ c_1 &= 0 \quad (2) \end{aligned}} \longrightarrow \boxed{\begin{aligned} &(2)\ c_1 = 0 \\ &(1)\ 0 + c_2 = 0 \\ &\quad c_2 = 0 \end{aligned}}$$

The only solution to

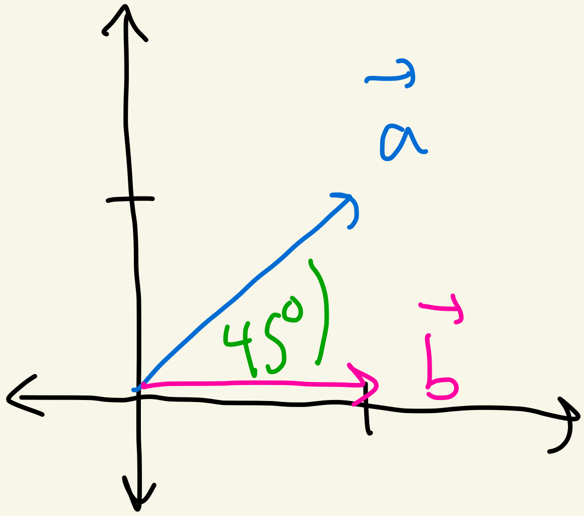
$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$.

Thus, $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle 1, 0 \rangle$ are linearly independent.

So, $\beta = [\vec{a}, \vec{b}]$ is a basis.

<u>Q</u>: Is $\beta$ an orthogonal basis?




$$\vec{a} \cdot \vec{b} = (1)(1) + (1)(0) = 1 \neq 0$$

No, $\beta$ is <u>not</u> an orthogonal basis.

<u>Q</u>: Is $\beta$ an orthonormal basis?

No, because $\beta$ is not an orthogonal basis.

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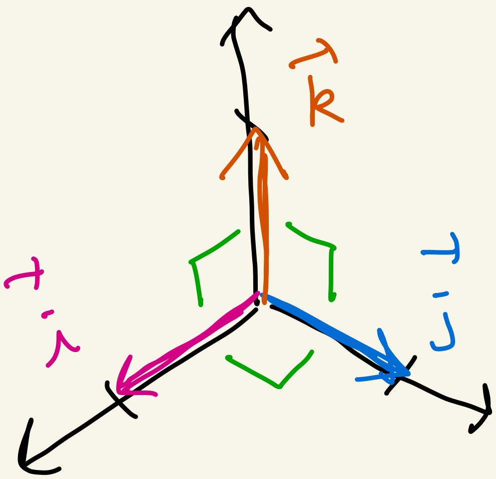
<u>Ex</u>: In $\mathbb{R}^3$, let $\beta = [\vec{i}, \vec{j}, \vec{k}]$ where $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$. We saw previously that $\beta$ is a basis for $\mathbb{R}^3$.

**Q:** Is $\beta$ an orthogonal basis?




$$\vec{i}\cdot\vec{k} = (1)(0)+(0)(0)+(0)(1) = 0$$
$$\vec{j}\cdot\vec{k} = (0)(0)+(1)(0)+(0)(1) = 0$$
$$\vec{i}\cdot\vec{j} = (1)(0)+(0)(1)+(0)(0) = 0$$

Yes, $\beta$ is an orthogonal basis.

**Q:** Is $\beta$ an orthonormal basis?

(i) $\beta$ is orthogonal

(ii) $\|\vec{i}\| = \sqrt{1^2+0^2+0^2} = 1$

$\|\vec{j}\| = \sqrt{0^2+1^2+0^2} = 1$

$\|\vec{k}\| = \sqrt{0^2+0^2+1^2} = 1$

So, yes, $\beta$ is an orthonormal basis.

## Coordinate dot-product theorem

Let $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$ be a basis for $\mathbb{R}^n$.

Let $\vec{v}$ be some vector in $\mathbb{R}^n$.

- If $\beta$ is an orthogonal basis, then

$$\vec{v} = \underbrace{\left(\frac{\vec{v}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\right)}\vec{v}_1 + \underbrace{\left(\frac{\vec{v}\cdot\vec{v}_2}{\|\vec{v}_2\|^2}\right)}\vec{v}_2 + \cdots + \underbrace{\left(\frac{\vec{v}\cdot\vec{v}_n}{\|\vec{v}_n\|^2}\right)}\vec{v}_n$$

$\vec{v}$'s $\beta$-coordinates

- If $\beta$ is an orthonormal basis, then

$$\vec{v} = \underbrace{(\vec{v}\cdot\vec{v}_1)}\vec{v}_1 + \underbrace{(\vec{v}\cdot\vec{v}_2)}\vec{v}_2 + \cdots + \underbrace{(\vec{v}\cdot\vec{v}_n)}\vec{v}_n$$

$\vec{v}$'s $\beta$-coordinates

Ex: In $\mathbb{R}^2$, consider the orthonormal basis $\beta = [\vec{i}, \vec{j}]$.
where $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$

Let $\vec{v} = \langle 7, -2 \rangle$.
Let's find $\underbrace{[\vec{v}]_\beta}_{\text{means: } \vec{v}\text{'s } \beta\text{-coordinates}}$

From the coordinate dot-product thm.

$$\vec{v} = (\vec{v} \cdot \vec{i})\vec{i} + (\vec{v} \cdot \vec{j})\vec{j}$$
$$= \left(\langle 7, -2 \rangle \cdot \langle 1, 0 \rangle\right)\vec{i} + \left(\langle 7, -2 \rangle \cdot \langle 0, 1 \rangle\right)\vec{j}$$
$$= \left((7)(1) + (-2)(0)\right)\vec{i} + \left((7)(0) + (-2)(1)\right)\vec{j}$$
$$= 7\vec{i} - 2\vec{j}$$

So, $[\vec{v}]_\beta = \langle 7, -2 \rangle$

Ex: In $\mathbb{R}^2$, let $\beta = [\vec{a}, \vec{b}]$ where $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle -1, 1 \rangle$. We saw this is an orthogonal basis.

Let $\vec{v} = \langle -8, 7 \rangle$.

Find $[\vec{v}]_\beta$ ← $\vec{v}$'s $\beta$-coordinates

From the coordinate dot-product thm:

$$\vec{v} = \left(\frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2}\right)\vec{a} + \left(\frac{\vec{v} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

$$= \left(\frac{(-8)(1) + (7)(1)}{\left(\sqrt{1^2 + 1^2}\right)^2}\right)\vec{a} + \left(\frac{(-8)(-1) + (7)(1)}{\left(\sqrt{(-1)^2 + 1^2}\right)^2}\right)\vec{b}$$

$$= -\frac{1}{2}\vec{a} + \frac{15}{2}\vec{b}$$ ← $\boxed{[\vec{v}]_\beta = \left\langle -\frac{1}{2}, \frac{15}{2} \right\rangle}$

Check: $-\frac{1}{2}\vec{a} + \frac{15}{2}\vec{b} = -\frac{1}{2}\langle 1, 1 \rangle + \frac{15}{2}\langle -1, 1 \rangle$

$$= \left\langle -\frac{1}{2} - \frac{15}{2}, -\frac{1}{2} + \frac{15}{2} \right\rangle$$

$$= \langle -8, 7 \rangle = \vec{v}$$

**Note:** To turn an orthogonal basis into an orthonormal basis, just divide each vector by its length.

**Ex:**

**orthogonal**

$\vec{a} = \langle 1, 1 \rangle$

$\vec{b} = \langle -1, 1 \rangle$

**orthonormal**

$$\frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{\sqrt{2}}\vec{a} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\frac{1}{\|\vec{b}\|}\vec{b} = \frac{1}{\sqrt{2}}\vec{b} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$