Math 2550 - 01

10/30/24

Topic 7 - Subspaces of $\mathbb{R}^n$

<u>Def:</u> Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ be r vectors in $\mathbb{R}^n$. The set of all linear combinations

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_r\vec{v}_r$$

is called the <u>subspace</u> spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$. We denote it by

$$\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r)$$
$$= \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_r\vec{v}_r \;\middle|\; c_1, c_2, \ldots, c_r \in \mathbb{R} \right\}$$

Call this subspace W. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are linearly independent then we say that the <u>dimension</u> of W is r and write $\dim(W) = r$. and call $\beta = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r]$ a <u>basis</u>

for W.

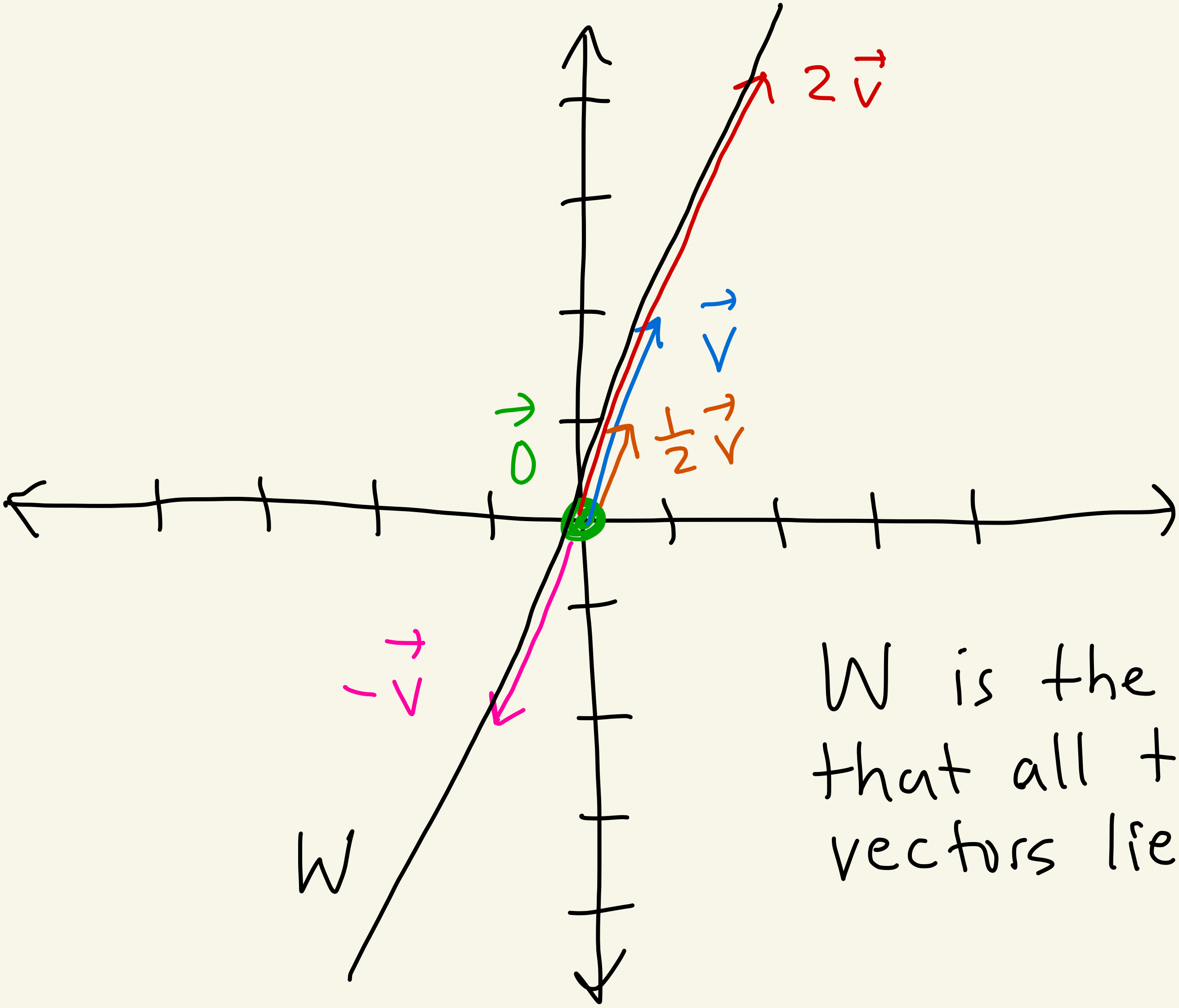
<u>Ex:</u> In $\mathbb{R}^2$, let $\vec{v} = \langle 1,2 \rangle$.

Let

$$W = \text{span}(\vec{v}) = \{ c\vec{v} \mid c \in \mathbb{R} \}$$
$$= \{ c\langle 1,2 \rangle \mid c \in \mathbb{R} \}$$

For example some vectors in W are:

$$2 \cdot \vec{v} = 2\langle 1,2 \rangle = \langle 2,4 \rangle$$
$$-\vec{v} = -\langle 1,2 \rangle = \langle -1,-2 \rangle$$
$$\frac{1}{2}\vec{v} = \frac{1}{2}\langle 1,2 \rangle = \langle \tfrac{1}{2},1 \rangle$$
$$1 \cdot \vec{v} = \langle 1,2 \rangle$$
$$0 \cdot \vec{v} = 0\langle 1,2 \rangle = \langle 0,0 \rangle = \vec{0}$$

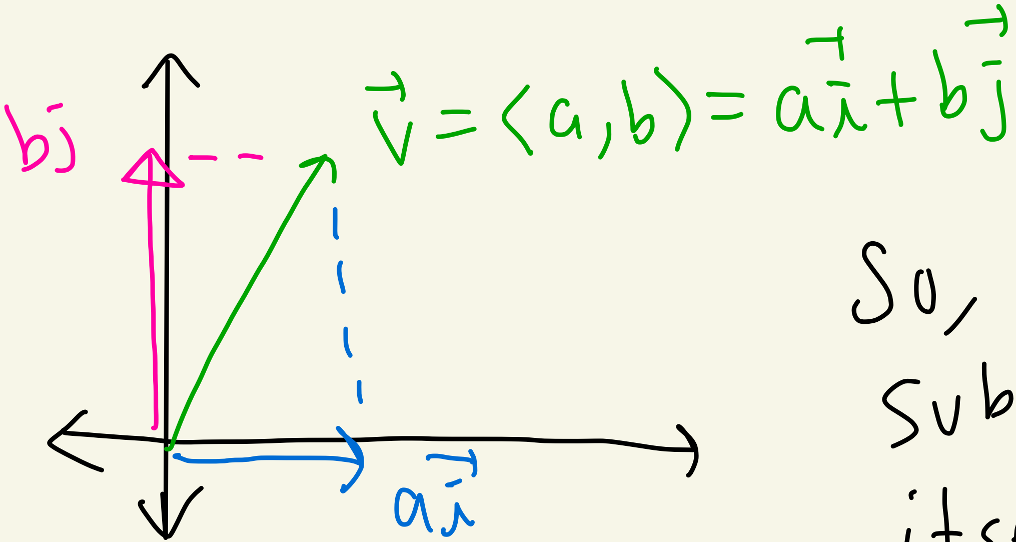


W is the line that all these vectors lie on

Since $\vec{v}$ is not the zero vector, $\beta = [\vec{v}]$ is a linearly independent set And $W = \text{span}(\vec{v})$. So, β is a basis for W. Since β has 1 vector, the dimension of W is $\dim(W) = 1$.

Ex: In $\mathbb{R}^2$, let $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$

We already know $\beta = [\vec{i}, \vec{j}]$ is a basis for all of $\mathbb{R}^2$, that is $\vec{i}, \vec{j}$ are linearly independent and $\text{span}(\vec{i}, \vec{j}) = \mathbb{R}^2$ because any vector $\vec{v} = \langle a, b \rangle = a\vec{i} + b\vec{j}$.



So, $\mathbb{R}^2$ is a subspace of itself.

And $\dim(\mathbb{R}^2) = 2$ because the basis β has 2 vectors in it.

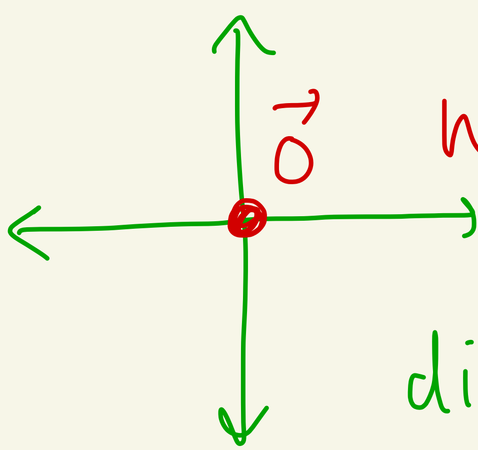
Def: In $\mathbb{R}^n$, let

$$W = \text{span}(\vec{0}) = \{c\vec{0} \mid c \in \mathbb{R}\} = \{\vec{0}\}$$

This is called the <u>trivial subspace</u>

It has no basis, however we just define the dimension to be 0.

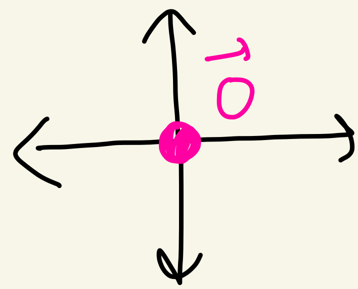
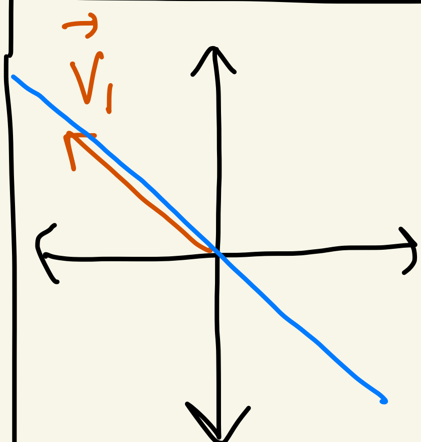
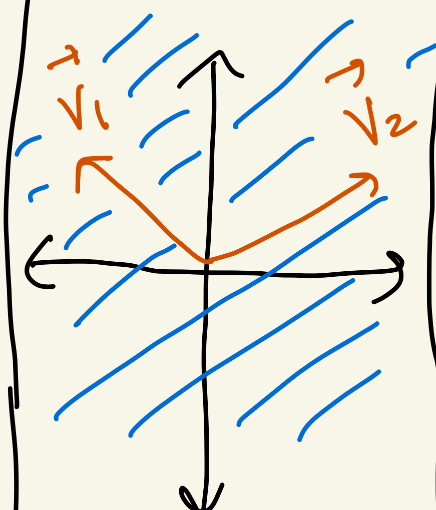
Ex: In $\mathbb{R}^2$:



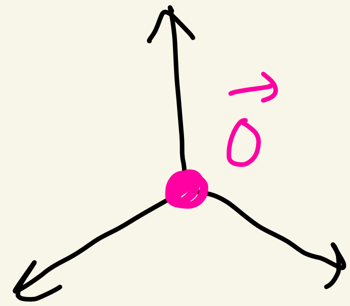
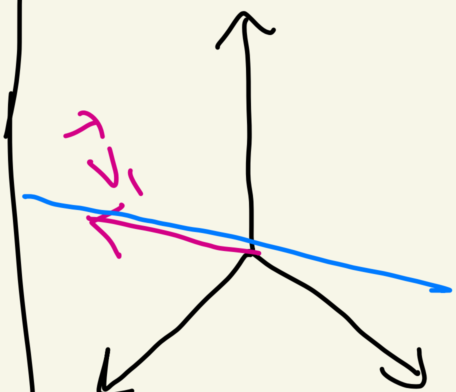
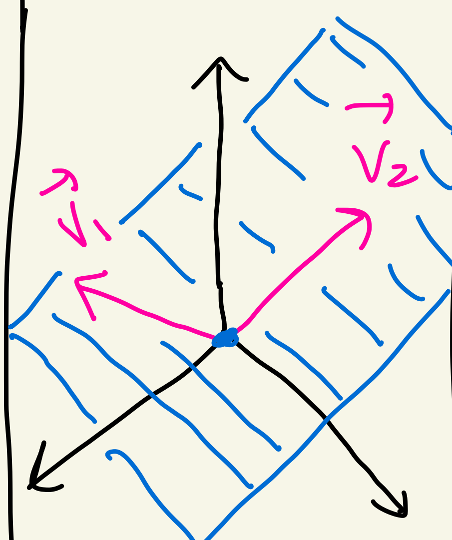
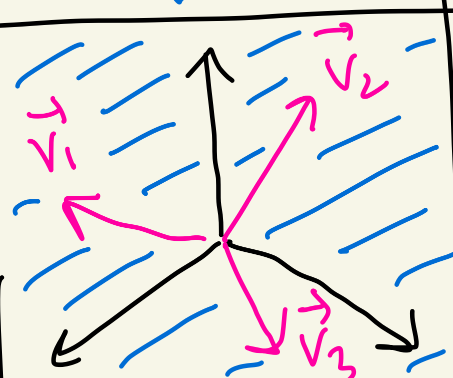
$W = \text{span}(\vec{0})$
$= \{\vec{0}\}$

$\dim(W) = 0$

All subspaces of $\mathbb{R}^2$

dimension r	basis of r linearly independent vectors	Picture of span of basis	description
0	no basis		the origin $\vec{0}$
1	$\vec{v}_1$		line through the origin
2	$\vec{v}_1, \vec{v}_2$		the entire plane $\mathbb{R}^2$

Subspaces in $\mathbb{R}^3$

dimension r	basis of r linearly indep. vectors	picture of span of basis	description
0	no basis		point at origin
1	$\vec{v}_1$		line through the origin
2	$\vec{v}_1, \vec{v}_2$		a plane through the origin that $\vec{v}_1, \vec{v}_2$ lie on
3	$\vec{v}_1, \vec{v}_2, \vec{v}_3$		all of $\mathbb{R}^3$

Homogeneous subspace theorem

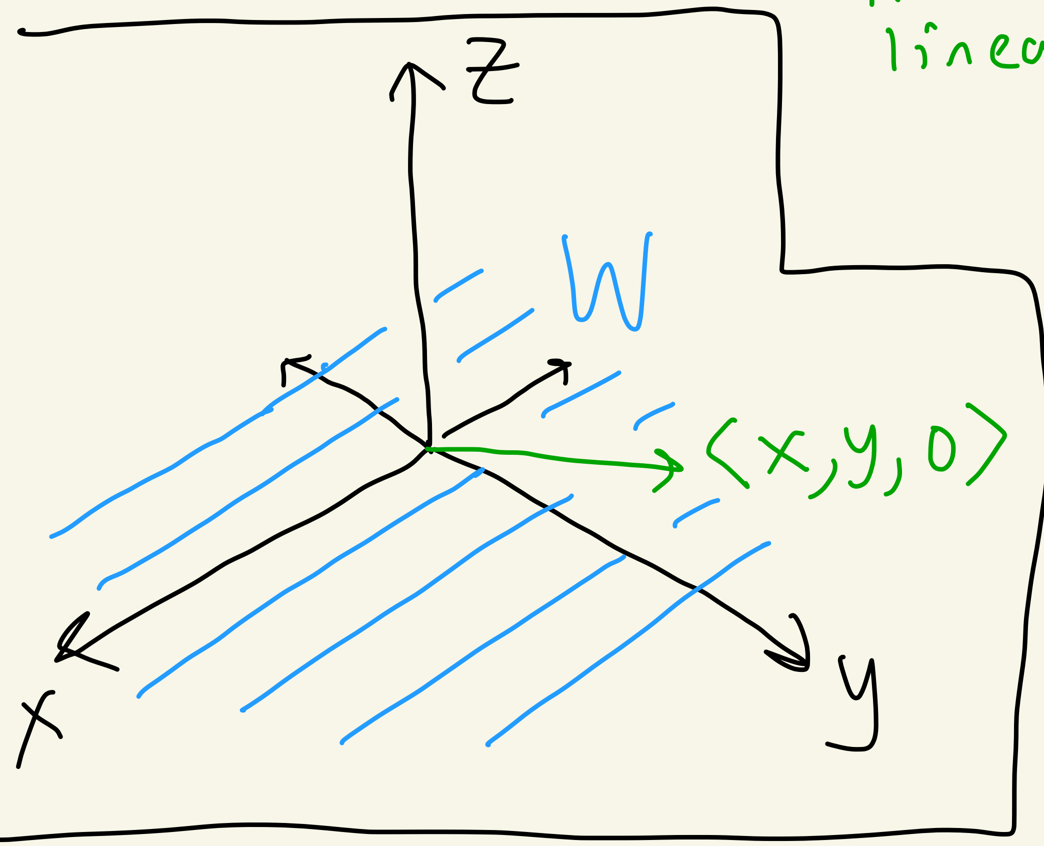
Let W be a subset of $\mathbb{R}^n$. Then W is a subspace if and only if W consists of all vectors $\vec{v} = \langle x_1, x_2, \dots, x_n \rangle$ that solve a homogeneous system of linear equations

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ &\;\,\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}}$$

homogeneous means $= 0$ on all equations

Ex: In $\mathbb{R}^3$, let

$$W = \{ \langle x, y, z \rangle \mid \underbrace{z = 0}_{\text{homogeneous linear system}} \}$$



W consists of all the vectors in the xy-plane.

Let's find a basis that spans W.

Suppose $\vec{v}$ is in W.

Then,

$$\vec{v} = \langle x, y, 0 \rangle = \langle x, 0, 0 \rangle + \langle 0, y, 0 \rangle$$

$$= x\langle 1, 0, 0 \rangle + y\langle 0, 1, 0 \rangle$$

$$= x\vec{\imath} + y\vec{\jmath}$$

So, $W = \text{span}(\vec{i}, \vec{j})$.

Are $\vec{i}, \vec{j}$ linearly independent?

Consider

$$c_1 \vec{i} + c_2 \vec{j} = \vec{0}$$

This becomes

$$c_1 \langle 1, 0, 0 \rangle + c_2 \langle 0, 1, 0 \rangle = \langle 0, 0, 0 \rangle$$

which gives

$$\langle c_1, 0, 0 \rangle + \langle 0, c_2, 0 \rangle = \langle 0, 0, 0 \rangle$$

$$\langle c_1, c_2, 0 \rangle = \langle 0, 0, 0 \rangle$$

Thus, $c_1 = 0$, $c_2 = 0$.

So the only solution to

$$c_1 \vec{i} + c_2 \vec{j} = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$. Thus, $\vec{i}, \vec{j}$ are linearly independent.

So, $\beta = [\vec{i}, \vec{j}]$ is a basis for W.

Thus, W has dimension 2 because β has 2 vectors in it.