2550-01
12/4/24

Q: List 2 elements from

$$S=\left\{c_1\langle 1,1,0\rangle + c_2\langle -2,0,1\rangle + c_3\langle 0,0,1\rangle \;\middle|\; c_1, c_2, c_3 \in \mathbb{R}\right\}$$

$c_1 = 1,\ c_2 = -2,\ c_3 = 0$

$$1\cdot\langle 1,1,0\rangle - 2\cdot\langle -2,0,1\rangle + 0\cdot\langle 0,0,1\rangle$$

$$= \langle 1,1,0\rangle + \langle 4,0,-2\rangle + \langle 0,0,0\rangle$$

$$= \boxed{\langle 5,1,-2\rangle} \leftarrow \text{a vector in } S$$

$c_1 = 1,\ c_2 = 1,\ c_3 = 1$

$$1\cdot\langle 1,1,0\rangle + 1\cdot\langle -2,0,1\rangle + 1\cdot\langle 0,0,1\rangle$$

$$= \boxed{\langle -1,1,2\rangle} \leftarrow \text{a vector in } S$$

T2 2 (modified)

GIVEN:

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Use A^{-1} to solve

$$\boxed{\begin{aligned} 2x + y &= 4 \\ 3x - y &= 7 \end{aligned}}$$

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} 2x+y \end{pmatrix}} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$(3x - y)$$

Multiply by A^{-1} to get

$$\underbrace{\begin{pmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4/5 + 7/5 \\ 12/5 - 14/5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11/5 \\ -2/5 \end{pmatrix}$$

$$\boxed{\text{Answer: } x = 11/5, \; y = -2/5}$$

$\boxed{T2}$

③ (modified)

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det\begin{pmatrix} -2 & 5 & 1 \\ 0 & -4 & 2 \\ -2 & 1 & 3 \end{pmatrix}$$

$$= (-2)\underbrace{\begin{vmatrix} -4 & 2 \\ 1 & 3 \end{vmatrix}}_{\begin{pmatrix} -2 & 5 & 1 \\ 0 & -4 & 2 \\ -2 & 1 & 3 \end{pmatrix}} - 0 + (-2)\underbrace{\begin{vmatrix} 5 & 1 \\ -4 & 2 \end{vmatrix}}_{\begin{pmatrix} -2 & 5 & 1 \\ 0 & -4 & 2 \\ -2 & 1 & 3 \end{pmatrix}}$$

$$= -2\left[(-4)(3) - (2)(1)\right] - 2\left[(5)(2) - (1)(-4)\right]$$

$$= -2\left[-14\right] - 2\left[14\right] = 0$$

Does A have an inverse? $\boxed{\text{NO}}$

T2

5 $\vec{a} = \langle 2, 1 \rangle$, $\vec{b} = \langle -1, 2 \rangle$

(a) Show $\vec{a}, \vec{b}$ are linearly independent.

$$c_1 \vec{a} + c_2 \vec{b} = \vec{0}$$

$$c_1 \langle 2, 1 \rangle + c_2 \langle -1, 2 \rangle = \langle 0, 0 \rangle$$

$$\langle 2c_1, c_1 \rangle + \langle -c_2, 2c_2 \rangle = \langle 0, 0 \rangle$$

$$\langle \underbrace{2c_1 - c_2}, \underbrace{c_1 + 2c_2} \rangle = \langle 0, 0 \rangle$$

$$\boxed{\begin{aligned} 2c_1 - c_2 &= 0 \\ c_1 + 2c_2 &= 0 \end{aligned}}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 2 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2\to R_2}\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}R_2\to R_2}\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

Gives: $\boxed{\begin{aligned} c_1+2c_2&=0 \quad (1)\\ c_2&=0 \quad (2)\end{aligned}}$

(2) $c_2=0$

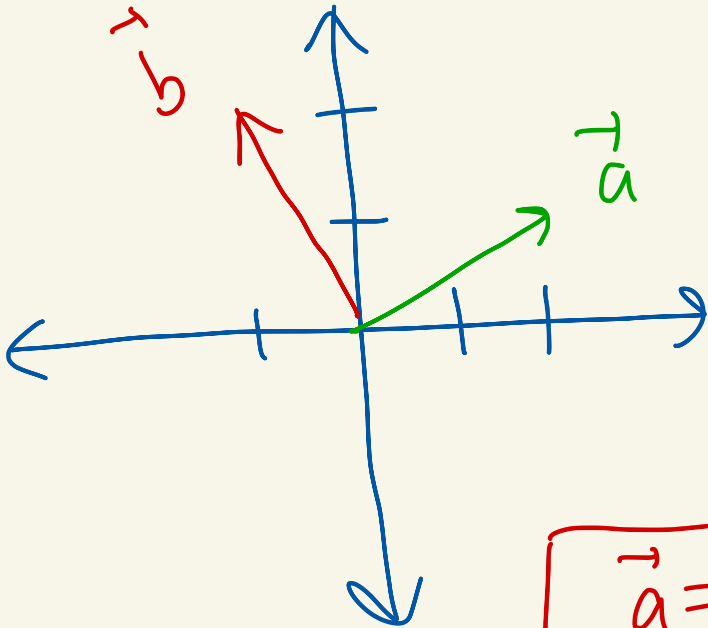
(1) $c_1=-2c_2=-2(0)=0$

Thus, the only solution to

$$c_1\vec{a}+c_2\vec{b}=\vec{0}$$

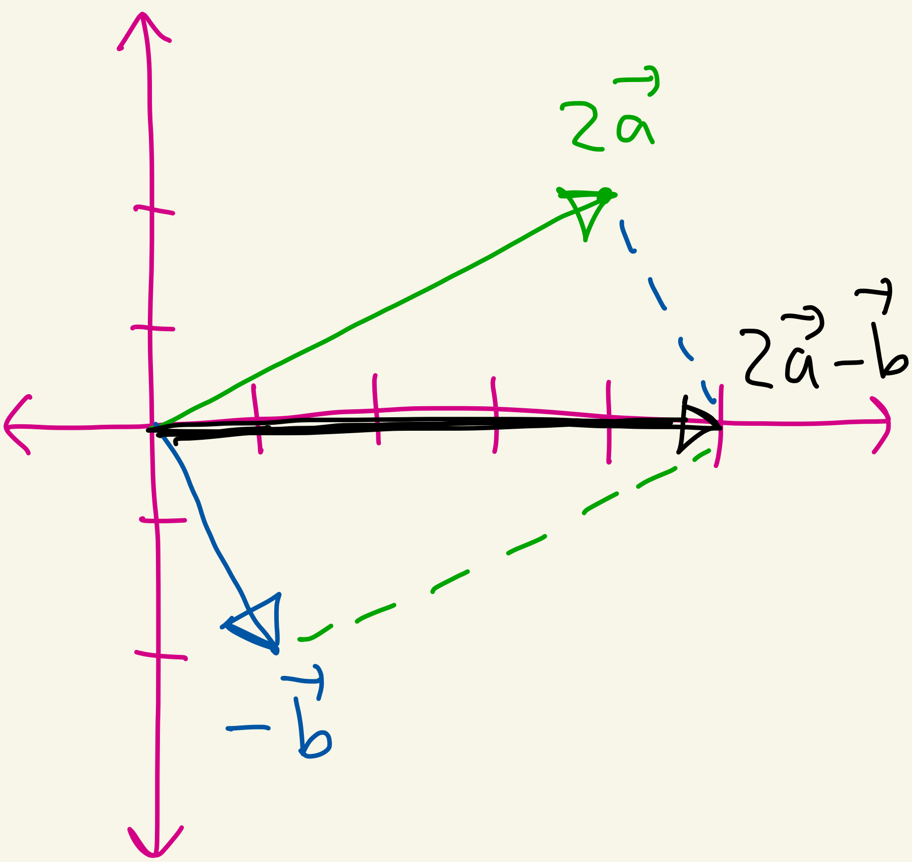
is $c_1=0, c_2=0$. Thus, $\vec{a},\vec{b}$ are linearly independent.

So, $\beta=[\vec{a},\vec{b}]$ is a basis for $\mathbb{R}^2$



$\boxed{\vec{a} = \langle 2, 1 \rangle, \quad \vec{b} = \langle -1, 2 \rangle}$

(b) Draw $2\vec{a}$, $-\vec{b}$, $2\vec{a} - \vec{b}$, and the parallelogram they make.



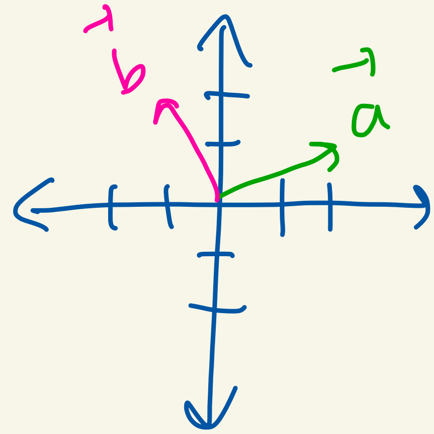
$$2\vec{a} = 2\langle 2, 1 \rangle = \langle 4, 2 \rangle$$

$$-\vec{b} = -\langle -1, 2 \rangle = \langle 1, -2 \rangle$$

$$2\vec{a} - \vec{b} = \langle 5, 0 \rangle$$

(c) $\beta = [\vec{a}, \vec{b}]$, $\vec{a} = \langle 2,1 \rangle$, $\vec{b} = \langle -1, 2 \rangle$

β is a basis by part (a)



Q1: Is β orthogonal?

Q2: Is β orthonormal?

Q1: $\vec{a} \cdot \vec{b} = \langle 2,1 \rangle \cdot \langle -1,2 \rangle$

$$= (2)(-1) + (1)(2)$$

$$= 0$$

So, β is an orthogonal basis.

Q2:
- β orthogonal ✓
- $\|\vec{a}\| = \sqrt{(2)^2 + (1)^2} = \sqrt{5} \neq 1$

  $\|\vec{b}\| = \sqrt{(-1)^2 + (2)^2} = \sqrt{5} \neq 1$

Need both $\|\vec{a}\| = 1$ & $\|\vec{b}\| = 1$ for β to be orthonormal.

So, β is <u>not</u> orthonormal.

(d) Use the coordinate dot product theorem to find $[\vec{v}]_\beta$ where $\vec{v} = \langle 4, -2 \rangle$

means: $\vec{v}$'s β-coordinates

$$\vec{v} = \left(\frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} \right) \vec{a} + \left(\frac{\vec{v} \cdot \vec{b}}{\|\vec{b}\|^2} \right) \vec{b}$$

$$= \left(\frac{\langle 4, -2 \rangle \cdot \langle 2, 1 \rangle}{(\sqrt{5})^2} \right) \vec{a} + \left(\frac{\langle 4, -2 \rangle \cdot \langle -1, 2 \rangle}{(\sqrt{5})^2} \right) \vec{b}$$

$$= \left(\frac{8 - 2}{5} \right) \vec{a} + \left(\frac{-4 - 4}{5} \right) \vec{b}$$

$$= \frac{6}{5} \vec{a} - \frac{8}{5} \vec{b}$$

So, $\vec{v} = \frac{6}{5}\vec{a} - \frac{8}{5}\vec{b}$

Thus, $[\vec{v}]_\beta = \langle \frac{6}{5}, -\frac{8}{5} \rangle$

(e) Suppose $[\vec{v}]_\beta = \langle 2, -3 \rangle$

Then, $\vec{v} = 2\vec{a} - 3\vec{b}$

$= 2\langle 2, 1 \rangle - 3\langle -1, 2 \rangle$

$= \langle 4, 2 \rangle + \langle 3, -6 \rangle$

$= \langle 7, -4 \rangle$