Math 2550-01

9/11/24

Def: A system of m linear equations in n unknowns $x_1, x_2, \dots, x_n$ is a list of m linear equations:

$$\boxed{\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}} \quad (*)$$

The solution space of $(*)$ consists of all $(x_1, x_2, \dots, x_n)$ that simultaneously solve all m equations, ie the common solutions to all m equations

The **augmented matrix** for (*) is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

x_1 column x_2 column x_n column   represents equal sign

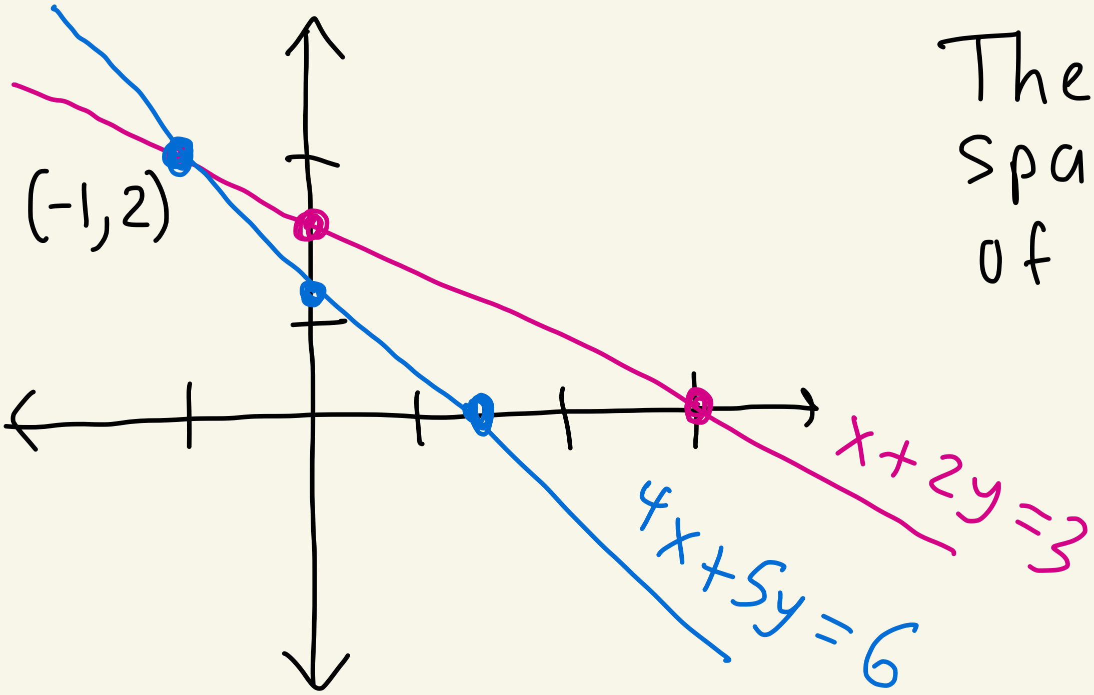
Ex:

$$\boxed{\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}}$$

$m = 2$ equations
$n = 2$ unknowns

Augmented matrix: $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right)$

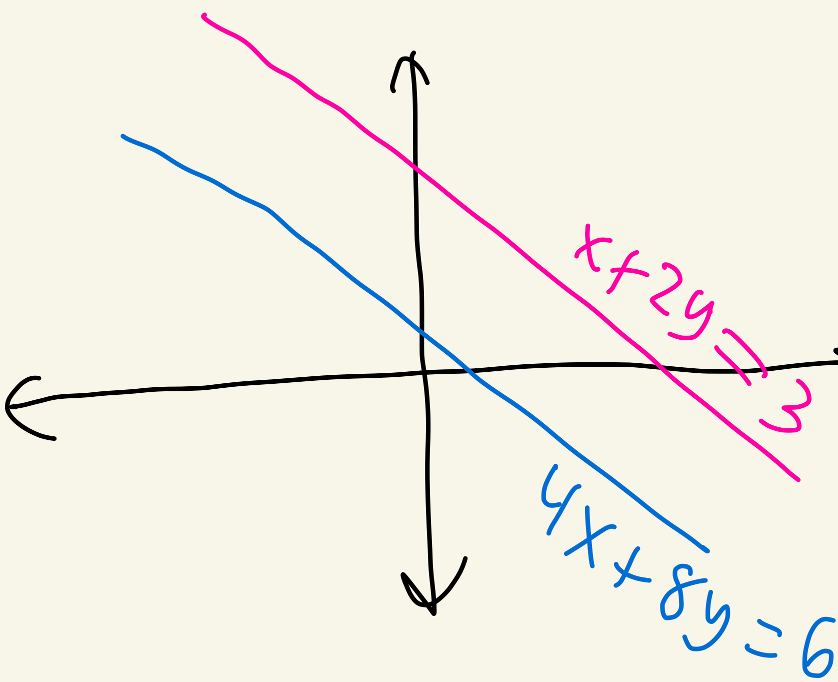
x column y column



The solution space consists of $(x,y)=(-1,2)$

Ex: $\boxed{\begin{aligned} x+2y&=3 \\ 4x+8y&=6 \end{aligned}}$ $m=2$ eqns. $n=2$ unknowns

Augmented matrix: $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 6 \end{array}\right)$



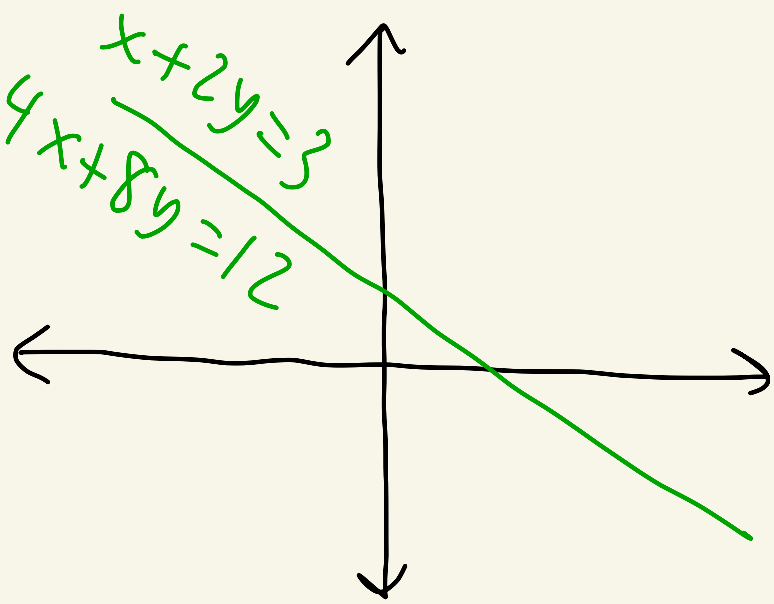
These lines are parallel. They don't intersect.

There are no solutions to the system

Ex: $$\boxed{\begin{aligned} x+2y&=3 \\ 4x+8y&=12 \end{aligned}}$$

$m=2$ eqns
$n=2$ unknowns

Augmented matrix: $$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 12 \end{array}\right)$$



It's the same line twice.
There are an infinite # of solutions, all the points (x,y) on the line.

Ex:

$$\boxed{\begin{aligned} x - 2y + z &= 5 \\ 10y - 3z &= 2 \\ 5x + y + \tfrac{1}{2}z &= 0 \end{aligned}}$$

$m = 3$ eqns

$n = 3$ unknowns

Augmented matrix

$$\begin{array}{cccc} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 10 & -3 & 2 \\ 5 & 1 & \tfrac{1}{2} & 0 \end{array}\right) \\ \begin{array}{ccc} x\text{ column} & y\text{ column} & z\text{ column} \end{array} \end{array}$$

<u>Def:</u> Given a system of linear equations there are three operations that we call <u>elementary row operations.</u>

They are:

1. Multiply one of the rows/equations by a non-zero number.
2. Interchange two rows/equations.
3. Add a multiple of one row/equation to another row/equation.

Ex: (multiply row/eqn. by non-zero number)

Equation viewpoint

$$\begin{aligned} 3x - 6y + 4z &= 1 \\ x + y \quad\quad &= 0 \\ 2x - y + z &= 5 \end{aligned} \xrightarrow{\frac{1}{3}R_1 \to R_1} \begin{aligned} x - 2y + \tfrac{4}{3}z &= \tfrac{1}{3} \\ x + y \quad\quad &= 0 \\ 2x - y + z &= 5 \end{aligned}$$

Matrix viewpoint

$$\left(\begin{array}{ccc|c} 3 & -6 & 4 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 5 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -2 & \frac{4}{3} & \frac{1}{3} \\ 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 5 \end{array}\right)$$

Ex: (Interchange two rows/equations)

Equation viewpoint

$$\begin{aligned} 2x - y &= 1 \\ -5x + 2y &= 0 \\ x - y &= 2 \end{aligned} \xrightarrow{R_1 \leftrightarrow R_3} \begin{aligned} x - y &= 2 \\ -5x + 2y &= 0 \\ 2x - y &= 1 \end{aligned}$$

Matrix viewpoint

$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ -5 & 2 & 0 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ -5 & 2 & 0 \\ 2 & -1 & 1 \end{array}\right)$$

Ex: (Add a multiple of one row/equation to another row/equation)

$\boxed{\text{Equation viewpoint}}$

$$\begin{array}{l} x+y-z=1 \\ 2x+y+2z=0 \\ x-y-z=3 \end{array} \xrightarrow{-2R_1+R_2 \to R_2} \begin{array}{l} x+y-z=1 \\ -y+4z=-2 \\ x-y-z=3 \end{array}$$

$$\begin{array}{rl} & -2x-2y+2z=-2 \leftarrow \boxed{-2R_1} \\ + & 2x+y+2z=0 \leftarrow \boxed{R_2} \\ \hline & -y+4z=-2 \leftarrow \boxed{\text{new } R_2} \end{array}$$

$\boxed{\text{Matrix viewpoint}}$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & -1 & -1 & 3 \end{array}\right) \xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & 4 & -2 \\ 1 & -1 & -1 & 3 \end{array}\right)$$

$$
\begin{array}{c}
\left(\begin{array}{ccc|c} -2 & -2 & 2 & -2 \end{array}\right) \leftarrow \boxed{-2R_1} \\
+\left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \end{array}\right) \leftarrow \boxed{R_2} \\
\hline
\left(\begin{array}{ccc|c} 0 & -1 & 4 & -2 \end{array}\right) \leftarrow \boxed{\text{new } R_2}
\end{array}
$$