Math 2550-01
9/23/24

Ex: Solve

$$\boxed{\begin{array}{rcl} a + 3b - 2c \phantom{{}-2d} + 2e \phantom{{}-3f} &=& 0 \\ 2a + 6b - 5c - 2d + 4e - 3f &=& -1 \\ 5c + 10d + 15f &=& 5 \\ 2a + 6b + 8d + 4e + 18f &=& 6 \end{array}}$$

We get:

already 1 here

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

use the 1 to make these 0.

$$\xrightarrow[-2R_1+R_4\to R_4]{-2R_1+R_2\to R_2}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

make this 1

$$\xrightarrow{-R_2\to R_2}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

Use the 1 to make these 0

$$\xrightarrow[-4R_2+R_4\to R_4]{-5R_2+R_3\to R_3}\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

make this 1

$$\xrightarrow{\frac{1}{6}R_3 \to R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

in row echelon form

We get:

$$\boxed{\begin{aligned} a + 3b - 2c \quad\quad + 2e \quad\quad &= 0 \\ c + 2d \quad\quad + 3f &= 1 \\ f &= 1/3 \\ 0 &= 0 \end{aligned}}$$

leading variables: a, c, f

free variables: b, d, e

Solve for leading variables and give the free variables a new name.

$$\boxed{\begin{aligned} a &= -3b+2c-2e \quad (1)\\ c &= 1-2d-3f \quad (2)\\ f &= 1/3 \quad (3)\\ b &= s \quad (4)\\ d &= t \quad (5)\\ e &= u \quad (6)\end{aligned}}$$

Back substitute:

(6) $e = u$

(5) $d = t$

④ $b = s$

③ $f = 1/3$

② $c = 1 - 2d - 3f = 1 - 2t - 3\left(\frac{1}{3}\right)$
$= -2t$

① $a = -3b + 2c - 2e$
$= -3s + 2(-2t) - 2u$
$= -3s - 4t - 2u$

$\boxed{\begin{array}{l} \underline{\text{Answer:}} \\ a = -3s - 4t - 2u \\ b = s \\ c = -2t \\ d = t \\ e = u \\ f = 1/3 \end{array} \quad \text{where } s, t, u \text{ can be any real numbers}}$

There's an infinite # of solutions to the system, one for each

value of s, t, u. For example, if $s=1, t=2, u=0$ then we get the following solution:
$(a,b,c,d,e,f) = (-11, 1, -4, 2, 0, 1/3)$

<u>Theorem</u>: A system of linear equations has either

(i) no solutions,

(ii) exactly one solution,

or (iii) infinitely many solutions

Topic 4 – The inverse of a matrix

<u>Def:</u> Let A be an $n \times n$ matrix. [So, A is a square matrix.] We say that A is <u>invertible</u> if there is an $n \times n$ matrix B where

$$AB = I_n = BA$$

So,
$AB = I_n$
$BA = I_n$

If this is the case, then we say that A and B are <u>inverses</u> of each other.

<u>Ex:</u> Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$. Let's show that A and B are inverses.

We have

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} (1 \ 1)\cdot\begin{pmatrix} -1 \\ 2 \end{pmatrix} & (1 \ 1)\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ (2 \ 1)\cdot\begin{pmatrix} -1 \\ 2 \end{pmatrix} & (2 \ 1)\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & 1-1 \\ -2+2 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Similarly

$$BA = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & -1+1 \\ 2-2 & 2-1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Since $AB = I_2 = BA$ we know that A and B are both invertible and are inverses of each other.