Math 2550-01
9/30/24

(Topic 4 continued...)

Last time we talked about writing a system in the form $A\vec{x} = \vec{b}$.

If A^{-1} exists we can solve for $\vec{x}$ like this:

$$A\vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I_n\vec{x} = A^{-1}\vec{b}$$

$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

Ex: Solve

$$\boxed{\begin{aligned} 3x \quad\quad\; &+ 3z = 9 \\ x + y &+ 2z = -4 \\ -2x + 3y \quad &\quad\;\; = 5 \end{aligned}} \qquad (*)$$

Write it as $A\vec{x} = \vec{b}$ as follows:

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

Side check: Multiply the left side of $A\vec{x} = \vec{b}$ above gives

$$\begin{pmatrix} 3x + 3z \\ x + y + 2z \\ -2x + 3y \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \longleftarrow \boxed{\text{same as } (*)}$$

We have

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

Last week we found A^{-1}. It is

$$A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$$

Multiply both sides of the above equation on the left by A^{-1} to get:

$$\underbrace{\underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_{A}}_{I_3} \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

We get

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \begin{pmatrix} (2)(9)+(-3)(-4)+(1)(5) \\ (\frac{4}{3})(9)+(-2)(-4)+(1)(5) \\ (-\frac{5}{3})(9)+(3)(-4)+(-1)(5) \end{pmatrix}$$

So, $\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \begin{pmatrix} 35 \\ 25 \\ -32 \end{pmatrix}$

Thus, (*) has one solution. It is
$x = 35$, $y = 25$, $z = -32$.

Topic 5 – Determinants

The determinant will allow us to detect when an $n \times n$ matrix has an inverse

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<u>Def:</u> Let A be an $n \times n$ matrix. Define A_{ij} to be the $(n-1) \times (n-1)$ matrix that is obtained by deleting row i and column j from A.

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} \longleftarrow \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \longleftarrow \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

<u>Def</u>: Let A be an $n \times n$ matrix. Let a_{ij} be the entry in row i and column j of A. The <u>determinant</u> of A, denoted by $\det(A)$, is defined as follows:

① If $n=1$ and $A=(a_{11})$ then $\det(A)=a_{11}$.

② If $n=2$ and $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

then $\underbrace{\det(A) = a_{11}a_{22} - a_{12}a_{21}}_{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}$

③ If $n \geq 3$, then pick any column j to "expand on" and define

$$\det(A) = \sum_{\underbrace{i=1}_{\substack{\text{sum over rows } i \\ \text{column } j \text{ is fixed}}}}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

Note: In step ③ you could also pick a row i to expand on. You'd get:

$$\det(A) = \underbrace{\sum_{j=1}^{n}}_{\substack{\text{sums over columns } j \\ \text{row } i \text{ is fixed}}} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{\bar{i}\bar{j}})$$

Note: It doesn't matter what row or column you pick in step 3. The answer in the end will be the same.

Notation: Another notation for determinant is using vertical bars like this:

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Ex: $\det(12) = 12$

Ex:

$$\det\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} = \underbrace{(1)(2) - (-1)(3)}_{\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}}$$

$$= 5$$

Ex: Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

Calculate $\det(A)$

Let's expand on column $j = 3$.

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\det(A) = \sum_{i=1}^{3} (-1)^{i+3} \cdot a_{i3} \cdot \det(A_{i3})$$

$$= (-1)^{1+3} \cdot a_{13} \cdot \det(A_{13}) \leftarrow \boxed{i=1}$$

$$+ (-1)^{2+3} \cdot a_{23} \cdot \det(A_{23}) \leftarrow \boxed{i=2}$$

$$+ (-1)^{3+3} \cdot a_{33} \cdot \det(A_{33}) \leftarrow \boxed{i=3}$$

$$= (1) \cdot (0) \cdot \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (-1) \cdot (3) \cdot \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (1) \cdot (-2) \cdot \begin{vmatrix} 3 & 1 \\ -2 & -4 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$= 0$$

$$- 3 \cdot \left[(3)(4) - (1)(5) \right]$$

$$- 2 \cdot \left[(3)(-4) - (1)(-2) \right]$$

$$= 0 - 3[7] - 2[-10]$$

$$= -1$$

So, $\det(A) = -1$.

PICTURE WAY FOR $(-1)^{i+j}$ term

$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

we used column $j = 3$