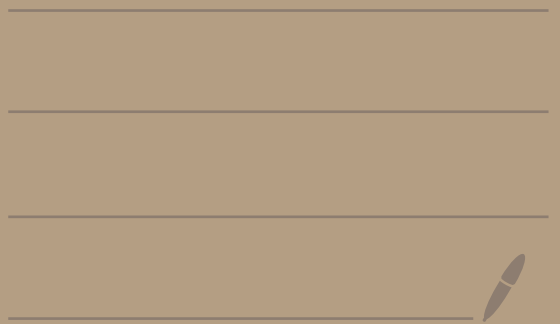


Math 2550-01

9/30/24



(Topic 4 continued...)

Last time we talked about
writing a system in the
form $A\vec{x} = \vec{b}$.

If A^{-1} exists we can solve
for \vec{x} like this:

$$A\vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I_n\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

Ex: Solve

$$\begin{cases} 3x & & + 3z = 9 \\ x + y & + 2z = -4 \\ -2x + 3y & & = 5 \end{cases} \quad (*)$$

Write it as $A\vec{x} = \vec{b}$ as follows:

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

Side check: Multiply the left side of $A\vec{x} = \vec{b}$ above gives

$$\begin{pmatrix} 3x + 3z \\ x + y + 2z \\ -2x + 3y \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

Same as
(*)

We have

$$\underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

Last week we found A^{-1} . It is

$$A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$$

Multiply both sides of the above equation on the left by A^{-1} to get:

$$\underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}}_A \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{\vec{b}}$$

I_3

We get

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \begin{pmatrix} (2)(9) + (-3)(-4) + (1)(5) \\ (\frac{4}{3})(9) + (-2)(-4) + (1)(5) \\ (-\frac{5}{3})(9) + (3)(-4) + (-1)(5) \end{pmatrix}$$

$$\text{So, } \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \begin{pmatrix} 35 \\ 25 \\ -32 \end{pmatrix}$$

Thus, (*) has one solution. It is
 $x = 35, y = 25, z = -32.$

Topic 5 - Determinants

The determinant will allow us to detect when an $n \times n$ matrix has an inverse

Def: Let A be an $n \times n$ matrix. Define $A_{\bar{i}\bar{j}}$ to be the $(n-1) \times (n-1)$ matrix that is obtained by deleting row \bar{i} and column \bar{j} from A .

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}$$

$$\leftarrow \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$\leftarrow \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

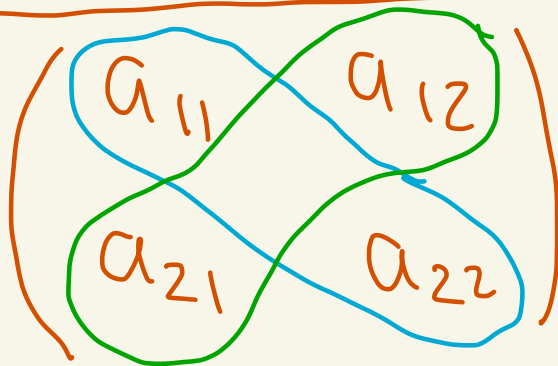
Def: Let A be an $n \times n$ matrix. Let a_{ij} be the entry in row i and column j of A . The determinant of A , denoted by $\det(A)$, is defined as follows:

① If $n=1$ and $A = (a_{11})$

then $\det(A) = a_{11}$.

② If $n=2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$



③ If $n \geq 3$, then pick any column j to "expand on" and define

$$\det(A) = \sum_{\bar{i}=1}^n (-1)^{\bar{i}+j} \cdot a_{\bar{i}j} \cdot \det(A_{\bar{i}j})$$

sum over rows \bar{i}
column j is fixed

Note: In step ③ you could also pick a row \bar{i} to expand on.

You'd get:

$$\det(A) = \sum_{\substack{j=1 \\ \text{sums over columns } j \\ \text{row } \bar{i} \text{ is fixed}}}^n (-1)^{\bar{i}+j} \cdot a_{\bar{i}j} \cdot \det(A_{\bar{i}j})$$

Note: It doesn't matter what row or column you pick in step 3. The answer in the end will be the same.

Notation: Another notation for determinant is using vertical bars like this:

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Ex: $\det(12) = 12$

Ex:

$$\det \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} = (1)(2) - (-1)(3)$$

$$\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

$$= 5$$

Ex: Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

Calculate $\det(A)$

Let's expand on

column $j=3$.

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\det(A) = \sum_{\bar{i}=1}^3 (-1)^{\bar{i}+3} \cdot a_{\bar{i}3} \cdot \det(A_{\bar{i}3})$$

$$= (-1)^{1+3} \cdot a_{13} \cdot \det(A_{13}) \leftarrow \boxed{\bar{i}=1}$$

$$+ (-1)^{2+3} \cdot a_{23} \cdot \det(A_{23}) \leftarrow \boxed{\bar{i}=2}$$

$$+ (-1)^{3+3} \cdot a_{33} \cdot \det(A_{33}) \leftarrow \boxed{\bar{i}=3}$$

$$= (1) \cdot (0) \cdot \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (-1) \cdot (3) \cdot \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$+ (1) \cdot (-2) \cdot \begin{vmatrix} 3 & 1 \\ -2 & -4 \end{vmatrix} \leftarrow \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$= 0$$

$$- 3 \cdot [(3)(4) - (1)(5)]$$

$$- 2 \cdot [(3)(-4) - (1)(-2)]$$

$$= 0 - 3[7] - 2[-10]$$

$$= -1$$

So, $\det(A) = -1$.

PICTURE WAY FOR $(-1)^{i+j}$ term

$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

↑
we
used
column
 $j = 3$