Math 2550-01
9/9/24

I'm redoing Topic 6 and
after on the website.
Both notes and HW

I'm keeping the old way I
did these later topics
at the bottom of the
website, but we won't use them.

<u>Ex:</u> Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Recall $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Then,

$$I_2 A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{2\times 2}$$

same ✓

answer is 2×2

$$= \begin{pmatrix} (1\ \ 0)\cdot\begin{pmatrix} 1 \\ 3 \end{pmatrix} & (1\ \ 0)\cdot\begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ (0\ \ 1)\cdot\begin{pmatrix} 1 \\ 3 \end{pmatrix} & (0\ \ 1)\cdot\begin{pmatrix} 2 \\ 4 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 1 + 0\cdot 3 & 1\cdot 2 + 0\cdot 4 \\ 0\cdot 1 + 1\cdot 3 & 0\cdot 2 + 1\cdot 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = A$$

So, $I_2 A = A$.

What about $A I_2$?

$$A I_2 = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{2 \times 2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{2 \times 2}$$

answer is 2×2

$$= \begin{pmatrix} 1\cdot 1 + 2 \cdot 0 & 1 \cdot 0 + 2 \cdot 1 \\ 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = A$$

Thus, $A I_2 = A$ and $I_2 A = A$.

Ex: Let

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

3×2 matrix

Recall

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then,

$$I_3 B = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{3 \times 3} \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}_{3 \times 2}$$

answer is 3×2

$$= \begin{pmatrix} (1\ 0\ 0) \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} & (1\ 0\ 0) \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \\ (0\ 1\ 0) \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} & (0\ 1\ 0) \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \\ (0\ 0\ 1) \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} & (0\ 0\ 1) \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = B.$$

So, $I_3 B = B$.

What about BI_3?

$$BI_3 = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}_{3\times 2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{3\times 3}$$

Undefined

not same

However,

$$BI_2 = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}_{3\times 2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{2\times 2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = B$$

$\checkmark$

you could do this step

So, $BI_2 = B$.

Algebraic properties of matrices

Let A, B, C be matrices.
Let α, β be real numbers.
Then the following are true
(where we assume the sizes of the matrices are such that the formulas are defined):

1. $A+B=B+A$
2. $A+(B+C)=(A+B)+C$
3. $A(BC)=(AB)C$
4. $A(B+C)=AB+AC$
5. $(B+C)A=BA+CA$
6. $A(B-C)=AB-AC$
7. $(B-C)A=BA-CA$
8. $\alpha(B+C)=\alpha B+\alpha C$

Note $AB=BA$ is <u>not</u> in this list. It's not always true

9. $\alpha(B-C) = \alpha B - \alpha C$
10. $(\alpha+\beta)A = \alpha A + \beta A$
11. $(\alpha-\beta)A = \alpha A - \beta A$
12. $\alpha(\beta A) = (\alpha\beta)A$
13. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
14. $(A^T)^T = A$
15. $(A+B)^T = A^T + B^T$
16. $(A-B)^T = A^T - B^T$
17. $(\alpha A)^T = \alpha A^T$
18. $(AB)^T = B^T A^T$ note order changes
19. If A is $m\times n$, then

$I_m A = A$ and $A I_n = A$.

(20) If A is $m \times n$, then

$$A - A = O_{m \times n}$$

and

$$A + O_{m \times n} = O_{m \times n} + A = A.$$

Let's prove (15) $(A+B)^T = A^T + B^T$ when A and B are both 2×2

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then the LHS gives

$$(A+B)^T = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right)^T$$

$$= \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}^T$$

$$= \begin{pmatrix} a+e & c+g \\ b+f & d+h \end{pmatrix}$$

Also, the RHS gives

$$A^T + B^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T + \begin{pmatrix} e & f \\ g & h \end{pmatrix}^T$$

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} + \begin{pmatrix} e & g \\ f & h \end{pmatrix}$$

$$= \begin{pmatrix} a+e & c+g \\ b+f & d+h \end{pmatrix}$$

EQUAL

Thus, $(A+B)^T = A^T + B^T$.

$\blacksquare$

Topic 3 - Systems of linear equations

<u>Def:</u> A <u>linear equation</u> in the n variables $x_1, x_2, \dots, x_n$ is an equation of the form

$$\boxed{a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b} \quad (*)$$

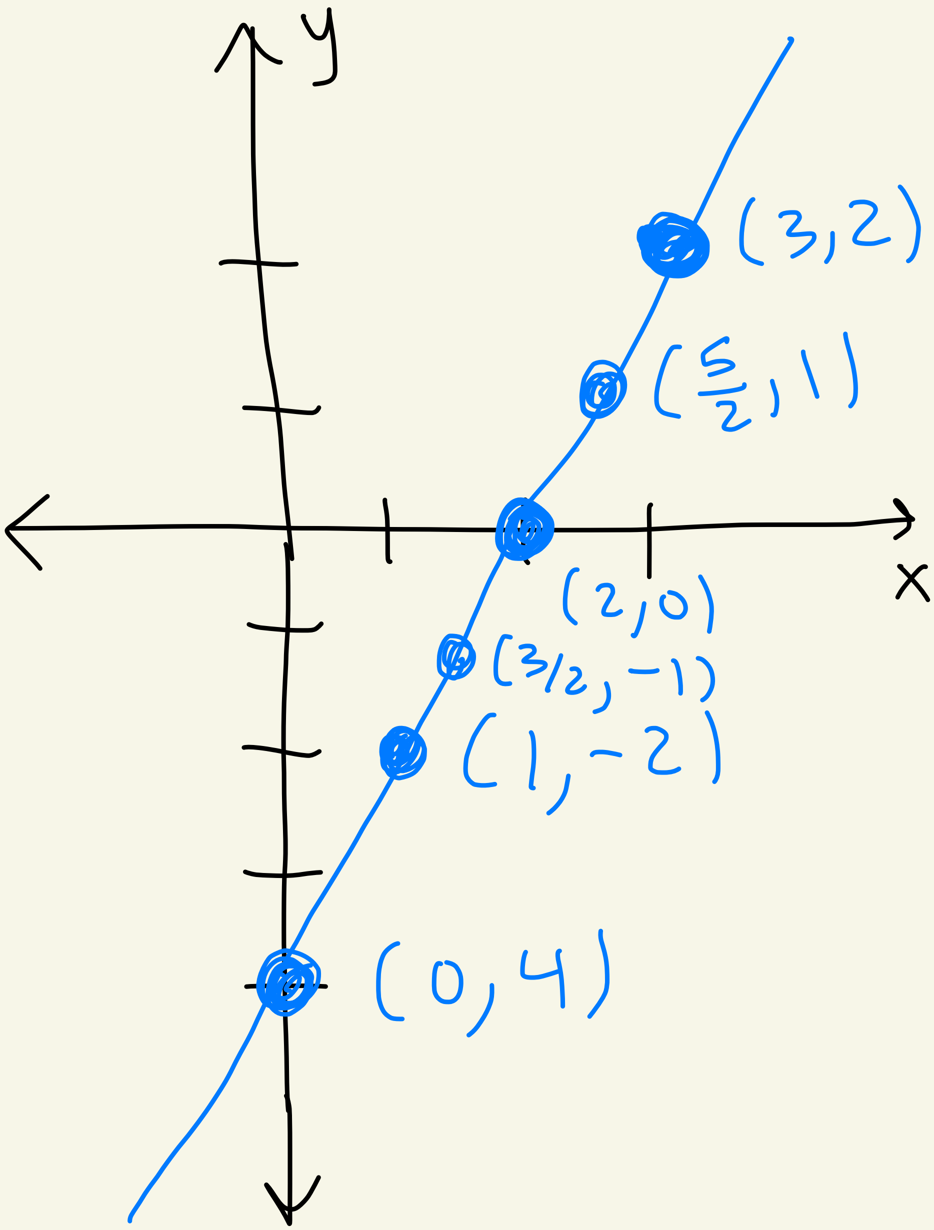
where $a_1, a_2, \dots, a_n, b$ are constant real numbers.

The <u>solution space</u> of $(*)$ consists of all $(x_1, x_2, \dots, x_n)$ that solve $(*)$.

Ex: Consider

$$4x - 2y = 8$$

linear equation
2 variables



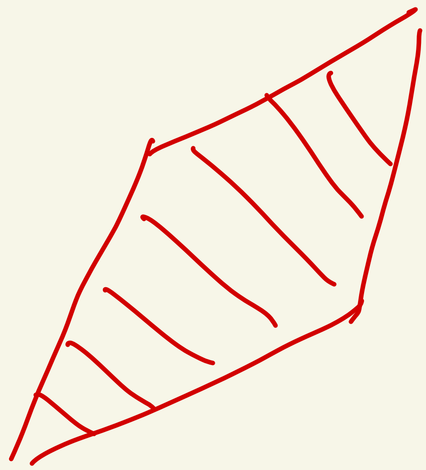
The points (x,y) on this line make the solution space to $4x-2y=8$

Ex: Consider

$$x + y - 2z = 3$$

linear equation
3 variables

In Calculus you learn this is the equation of a plane in 3d



Some points in the solution space are:

$$(x,y,z) = (1,0,-1), (2,1,0), (0,0,-\tfrac{3}{2}), \ldots$$

Some linear equations:

$$2x - \frac{1}{2}y + z - 3w + t = 2$$

$$x_1 - 2x_2 + x_3 + 100x_4 = 6$$

Some non-linear equations:

$$2x^2 + y = 6$$

$$5x + \sqrt{y} + z = 10$$

$$51\sin(x) + z^2 = x^3$$