Math 3450

1/30/24

Ex: Let C, D, E be sets.

Prove that

$$C \cap (D \cup E) = (C \cap D) \cup (C \cap E)$$

proof:

($\subseteq$) Let's show $C \cap (D \cup E) \subseteq (C \cap D) \cup (C \cap E)$

Let $x \in C \cap (D \cup E)$

Thus, $x \in C$ and $x \in D \cup E$.

Hence, $x \in C$ and $(x \in D$ or $x \in E)$.

So we have two cases:

(i) $x \in C$ and $x \in D$

or (ii) $x \in C$ and $x \in E$

case (i): Suppose $x \in C$ and $x \in D$

Then $x \in C \cap D$.

So, $x \in C \cap D$ or $x \in C \cap E$.

Thus, $x \in (C \cap D) \cup (C \cap E)$.

<u>case (ii)</u>: Suppose $x \in C$ and $x \in E$.

Then $x \in C \cap E$.

So $x \in C \cap D$ or $x \in C \cap E$.

Thus $x \in (C \cap D) \cup (C \cap E)$.

Thus in either case $x \in (C \cap D) \cup (C \cap E)$.

So, $C \cap (D \cup E) \subseteq (C \cap D) \cup (C \cap E)$.

($\supseteq$) Let's show $(C \cap D) \cup (C \cap E) \subseteq C \cap (D \cup E)$

Let $y \in (C \cap D) \cup (C \cap E)$.

Then $y \in C \cap D$ or $y \in C \cap E$.

<u>case (a)</u>: Suppose $y \in C \cap D$.

Then, $y \in C$ and $y \in D$.

So, $y \in C$ and $y \in D \cup E$.
Ergo, $y \in C \cap (D \cup E)$.

<u>case (b):</u> Suppose $y \in C \cap E$.
Then $y \in C$ and $y \in E$.
So, $y \in C$ and $y \in D \cup E$.
Thus, $y \in C \cap (D \cup E)$.

In either case, $y \in C \cap (D \cup E)$.
Thus, we have shown that
$$(C \cap D) \cup (C \cap E) \subseteq C \cap (D \cup E).$$

Since we have shown that
$$C \cap (D \cup E) \subseteq (C \cap D) \cup (C \cap E)$$
and $(C \cap D) \cup (C \cap E) \subseteq C \cap (D \cup E)$
we know $C \cap (D \cup E) = (C \cap D) \cup (C \cap E)$

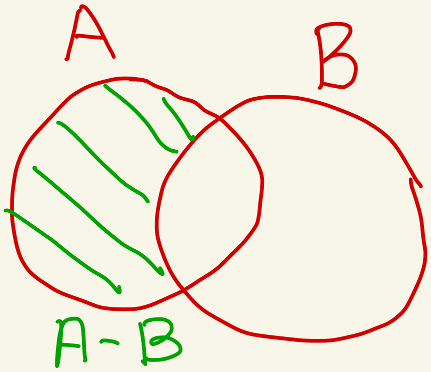
$\blacksquare$

Def: Let A and B be sets. We say that A and B are disjoint if $A \cap B = \phi$ where ϕ is the empty set.

Ex: $A = \{1, 2\}$ $\quad$ $A \cap B = \phi$

$B = \{3, 4\}$ $\quad$ So, A and B are disjoint

Def: Let A and B be sets. The difference of A and B is



$$A - B = \underbrace{\{x \mid x \in A \text{ and } x \notin B\}}$$

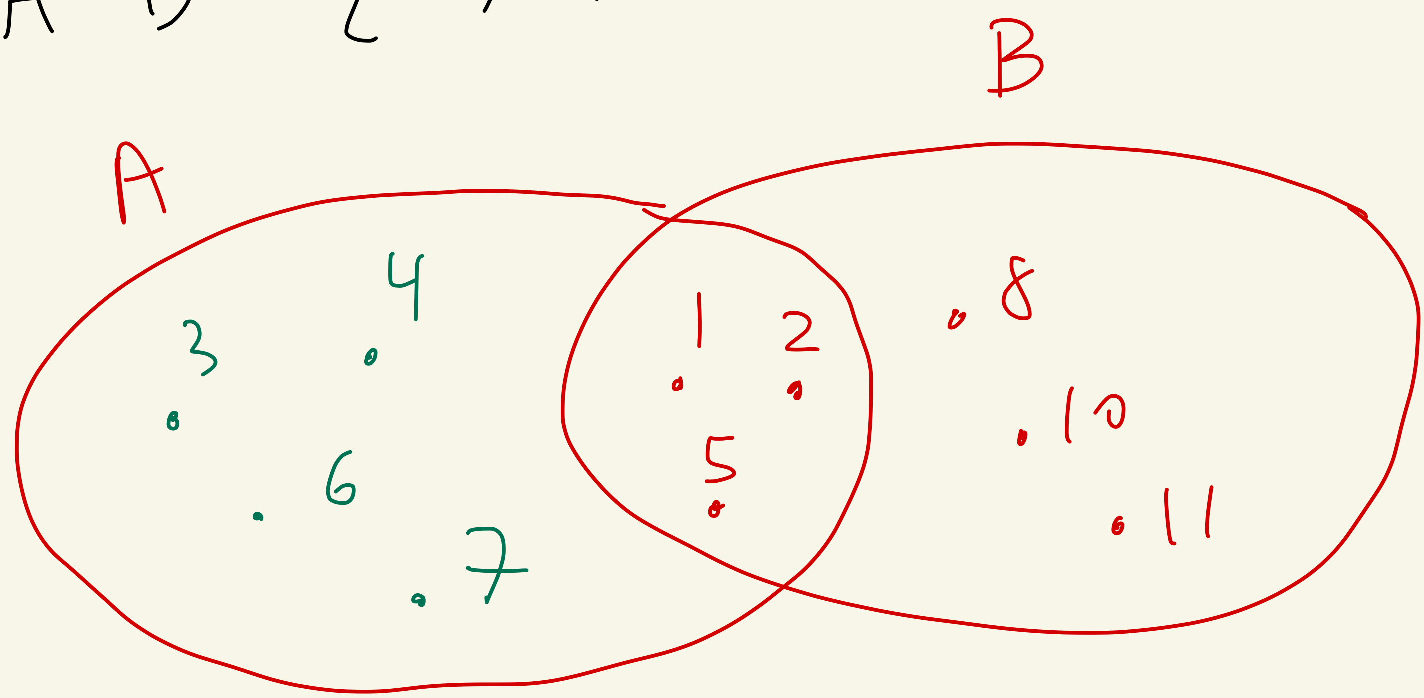
read: "all x where x is in A and x is not in B"

Notation: Some people write $A \setminus B$ for $A - B$.

Ex: $A = \{1, 2, 3, 4, 5, 6, 7\}$

$B = \{8, 10, 11, 2, 5, 1\}$

$A - B = \{3, 4, 6, 7\}$



$B - A = \{8, 10, 11\}$

$$A - \{10, 11, 20\} = A$$

$A = \{1, 2, 3, 4, 5, 6, 7\}$

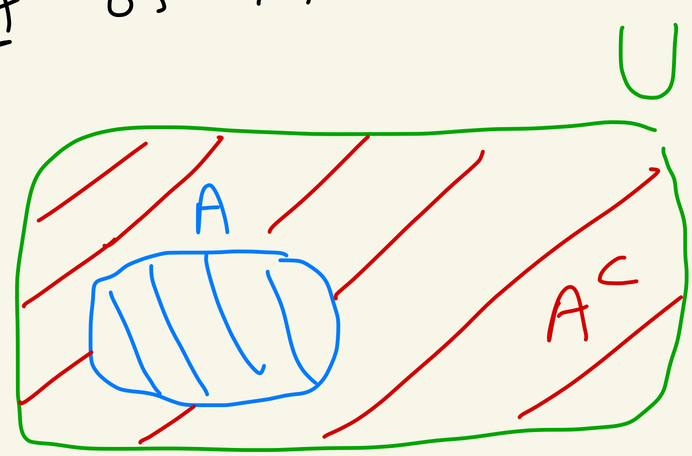
$$A - A = \emptyset$$

nothing left

- Sometimes all the sets you are looking at live inside of one big set. Let's call that big set a "universal set" or "universe"

Def: Let A be a set where U is a universal set (So, $A \subseteq U$). Then the complement of A with respect to U is

$$A^c = U - A$$



$= \{x \mid x \in U \text{ and } x \notin A\}.$

<u>Ex:</u>

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$A = \{2, 4, 6, 8, 10, 12\}$

$A^c = U - A = \{1, 3, 5, 7, 9, 11\}$

<u>Theorem:</u> (de Morgan's laws)

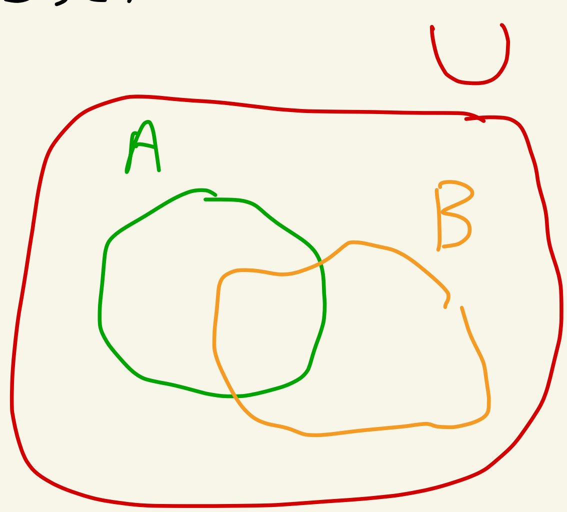
Let U be a universal set.

Let A and B be subsets of U.

Then:

① $(A \cup B)^c = A^c \cap B^c$

② $(A \cap B)^c = A^c \cup B^c$



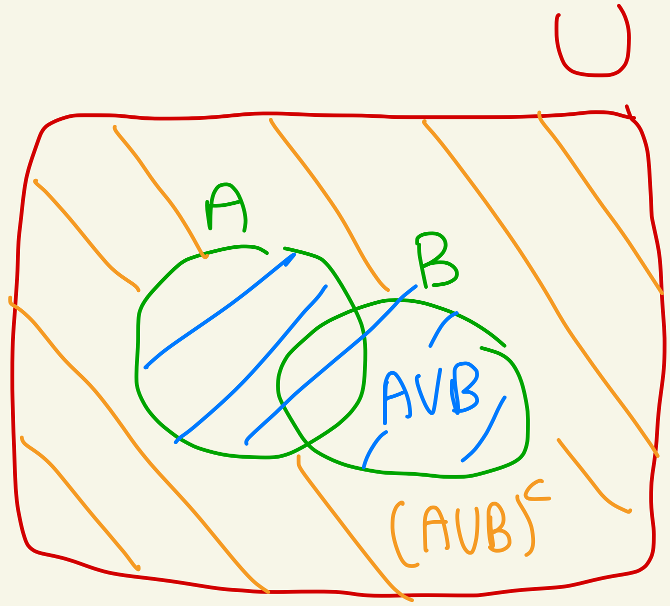
proof:

Let's prove ①. You can try ②.

Let's show $(A \cup B)^c \subseteq A^c \cap B^c$.

$\boxed{\subseteq}$ Let's show $(A \cup B)^c \subseteq A^c \cap B^c$.

Let $x \in (A \cup B)^c$.



Then, $x \in U$

and $x \notin A \cup B$.

So, $x \in U$ and

"$x \in A \cup B$" is not true.

So, $x \in U$ and "$x \in A$ or $x \in B$" is not true.

So, $x \in U$ and $x \notin A$ and $x \notin B$. $\boxed{2450}$

$\neg (P \text{ or } Q)$

$(\neg P) \text{ and } (\neg Q)$ equivalent

$\neg$ means not

Thus, $x \in A^c$

and $x \in B^c$.

So, $x \in A^c \cap B^c$.

Therefore, $(A \cup B)^c \subseteq A^c \cap B^c$.

$\boxed{\supseteq}$: Now let's show $A^c \cap B^c \subseteq (A \cup B)^c$.

Let $y \in A^c \cap B^c$.

So, $y \in A^c$ and $y \in B^c$.

So, $y \in U$ and $y \notin A$ and $y \notin B$.

Thus, $y \in U$ and $y \notin A \cup B$

same logic as above

So, $y \in (A \cup B)^c$.

Thus, $A^c \cap B^c \subseteq (A \cup B)^c$.

By $\boxed{\subseteq}$ and $\boxed{\supseteq}$ we have $(A \cup B)^c = A^c \cap B^c$

$\blacksquare$

Another way to prove:

$x \in (A \cup B)^c$

iff $x \in U$ and $x \notin A \cup B$

iff $x \in U$ and $x \notin A$ and $x \notin B$

iff $x \in A^c$ and $x \in B^c$

iff $x \in A^c \cap B^c$.

Thus, $(A \cup B)^c = A^c \cap B^c$. $\blacksquare$

$\boxed{\text{Workshop}}$

$\boxed{\text{Hw 2 - \#2}}$

$A = \{2k \mid k \in \mathbb{Z}\}$
$B = \{3n \mid n \in \mathbb{Z}\}$
Show $A \cap B = \{6l \mid l \in \mathbb{Z}\}$.

<u>proof:</u>

$\boxed{\subseteq}$: Let's show $A \cap B \subseteq \{6l \mid l \in \mathbb{Z}\}$.

Pick some $x \in A \cap B$.

Then, $x \in A$ and $x \in B$.

So, $x = 2k$ and $x = 3n$ where k, n are integers.

So, $2k = 3n$.

Thus, $3n$ is even.

We can't have n being odd since then $3n$ would be odd.
(odd * odd = odd)

So n is even.

Thus, $n = 2m$ where m is an integer.

So, $x = 3n = 3(2m) = 6m$

So, $x \in \{6l \mid l \in \mathbb{Z}\}$.

Thus, $A \cap B \subseteq \{6l \mid l \in \mathbb{Z}\}$.

$\boxed{\supseteq}$: Now let's show $\{6l \mid l \in \mathbb{Z}\} \subseteq A \cap B$.

Let $x \in \{6l \mid l \in \mathbb{Z}\}$.

Then $x = 6j$ where $j \in \mathbb{Z}$.

Thus, $x = 2(3j) \in A$.

And, $x = 3(2j) \in B$

So, $x \in A \cap B$.

Thus, $\{6l \mid l \in \mathbb{Z}\} \subseteq A \cap B$. $\blacksquare$