Math 3450
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<u>Recall:</u> Let $n, x \in \mathbb{Z}$ with $n \geq 2$.

Then,

$$\overline{x} = \{ y \mid y \in \mathbb{Z} \text{ and } \underbrace{y \equiv x \pmod{n}}_{\substack{\text{means:} \\ n \mid (y-x)}} \}$$

$\mathbb{Z}_n \leftarrow$ set of equivalence classed modulo n

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<u>Ex:</u> $n=3$

$$\overline{0} = \{ y \mid y \in \mathbb{Z},\ y \equiv 0 \pmod{3} \}$$
$$= \{ \ldots, -9, -6, -3, 0, 3, 6, 9, \ldots \}$$

$$\overline{1} = \{ y \mid y \in \mathbb{Z},\ y \equiv 1 \pmod{3} \}$$
$$= \{ \ldots, -8, -5, -2, 1, 4, 7, 10, \ldots \}$$

$\overline{2} = \{ y \mid y \in \mathbb{Z},\ y \equiv 2 \pmod 3 \}$

$= \{ \ldots, -7, -4, -1, 2, 5, 8, 11, \ldots \}$

$\mathbb{Z}_3 = \{ \overline{0}, \overline{1}, \overline{2} \}$

<u>Theorem:</u> (Equivalence classes modulo n)

Let $n \in \mathbb{Z}$ with $n \geq 2$.

Then

$$\mathbb{Z}_n = \{ \overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1} \}$$

These elements are all distinct.

That is, if $0 \leq x \leq y \leq n-1$ and $\overline{x} = \overline{y}$, then $x = y$.

proof: Let

$S = \{ \bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1} \}$.

We want to show that

$\mathbb{Z}_n = S$.

Note that $S \subseteq \mathbb{Z}_n$ because it consists of equivalence classes modulo n.

We just need to show that $\mathbb{Z}_n \subseteq S$.

Let $\bar{z} \in \mathbb{Z}_n$ where $z \in \mathbb{Z}$.

Divide z by n to get

$z = nq + r$

where $q, r \in \mathbb{Z}$ and $\underbrace{0 \le r < n}_{\text{same as } 0 \le r \le n-1}$

Then, $z - r = nq$.

So, $n \mid (z-r)$.

Thus, $z \equiv r \pmod{n}$.

Hence, $\overline{z} = \overline{r}$.

Thus, $\overline{z} \in S = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$

because $0 \le r \le n-1$.

Hence $\mathbb{Z}_n \subseteq S$.

So, $\mathbb{Z}_n = S$.

Why are all the elements

of $\{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$ distinct?

Suppose $\boxed{0 \leq x \leq y \leq n-1}$

with $\boxed{\bar{x} = \bar{y}}$.

Let's show this implies $x = y$.

Since $\bar{x} = \bar{y}$ we know that $x \equiv y \pmod{n}$.

super duper equiv. rel. thm. $\bar{x} = \bar{y}$ iff $x \sim y$

Thus, $n \mid (y - x)$.

Hence $\boxed{y - x = nk}$ for some $k \in \mathbb{Z}$.

Note $0 \leq y - x$ from above and $n \geq 2 > 0$, thus $k \geq 0$.

Since $x \leq y \leq n-1$ by

subtracting x we get

$$0 \le y - x \le n - 1 - x.$$

Since $0 \le x$ we know

$$n - 1 - x < n.$$

Thus, $\boxed{0 \le y - x < n}$

Summary so far:

$$\boxed{\begin{array}{l} y - x = nk \text{ with } k \geq 0 \\ \text{and } 0 \le y - x < n \end{array}}$$

Let's show $k = 0$.

Suppose instead that $k > 0$.

If so, then

$$0 \le y - x < n \le nk = y - x$$

assuming $k > 0$
ie $k \geq 1$

But then $y - x < y - x$
which can't happen.

Hence $k = 0$.

So, $y - x = nk = n(0) = 0$.

Thus, $y = x$.

$\boxed{\text{FINITO}}$

Ex:

$$\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$$

$$\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$$

$$\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$$

$$\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$$

HW 3

9 Let

$S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$

$= \{(2,-1), (-3,5), (10,21), \dots\}$

think $\frac{2}{-1}$ think $\frac{-3}{5}$ think $\frac{10}{21}$

Define $(a,b) \sim (c,d)$ to mean $ad = bc$.

idea: $\frac{a}{b} = \frac{c}{d}$ iff $ad = bc$

(a) Is $(1,5) \sim (-3,-15)$?

Check: $(1)(-15) = (5)(-3)$ ✓

Yes, $(1,5) \sim (-3,-15)$

(b) Is $(-1,1) \sim (2,3)$?

No because $(-1)(3) \neq (1)(2)$.

(c) Prove that $\sim$ is an equivalence relation on S.

$\boxed{\text{proof:}}$

(reflexive)

Let $(a,b) \in S$.

Then, $(a,b) \sim (a,b)$ because $ab = ba$.

(symmetric)

Let $(a,b),(c,d) \in S$.

Suppose $(a,b) \sim (c,d)$.

Then, $ad = bc$.

Thus, $cb = da$.

So, $(c,d) \sim (a,b)$.

(transitivity)

Let $(a,b),(c,d),(e,f) \in S$.

Then, $b \neq 0$, $d \neq 0$, $f \neq 0$.

Suppose $(a,b) \sim (c,d)$

and $(c,d) \sim (e,f)$.

Then, $ad = bc$ and $cf = de$.

Hence,

$$ad = bc = \underbrace{b\left(\frac{de}{f}\right)}_{\text{ok since } f \neq 0} = \frac{bde}{f}.$$

Since $d \neq 0$ we can divide by d to get $a = \frac{be}{f}$.

Thus, $af = be$.
Hence $(a,b) \sim (e,f)$.

$\blacksquare$