Math 3450
2/6/24

Def: Let $\mathcal{A}$ be a non-empty family of sets.

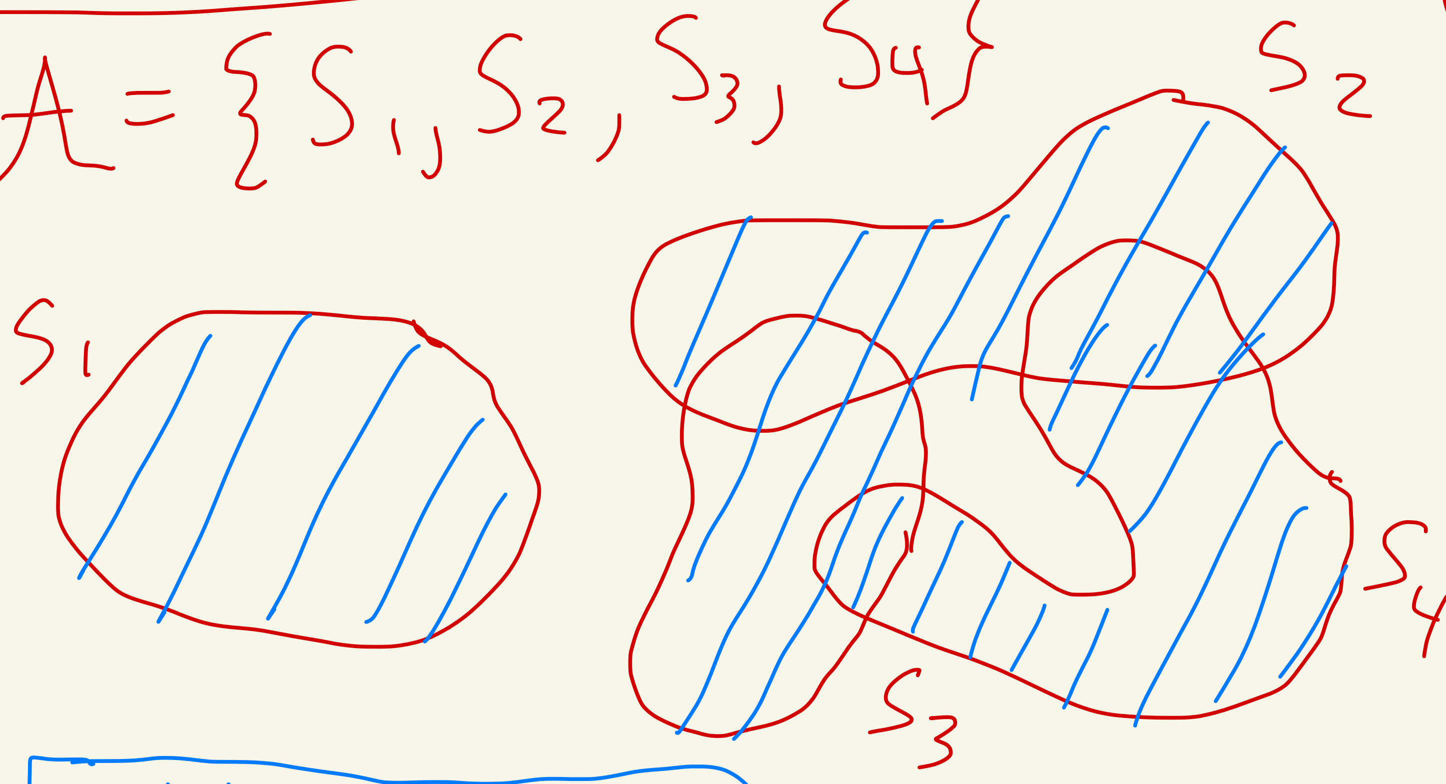
Define the union over $\mathcal{A}$ to be

$$\bigcup_{S\in\mathcal{A}} S = \{x \mid x \in S \text{ for some } S \in \mathcal{A}\}$$

$$= \{x \mid \text{there exists some } S\in\mathcal{A} \text{ where } x \in S\}$$

Ex:

$\mathcal{A} = \{S_1, S_2, S_3, S_4\}$

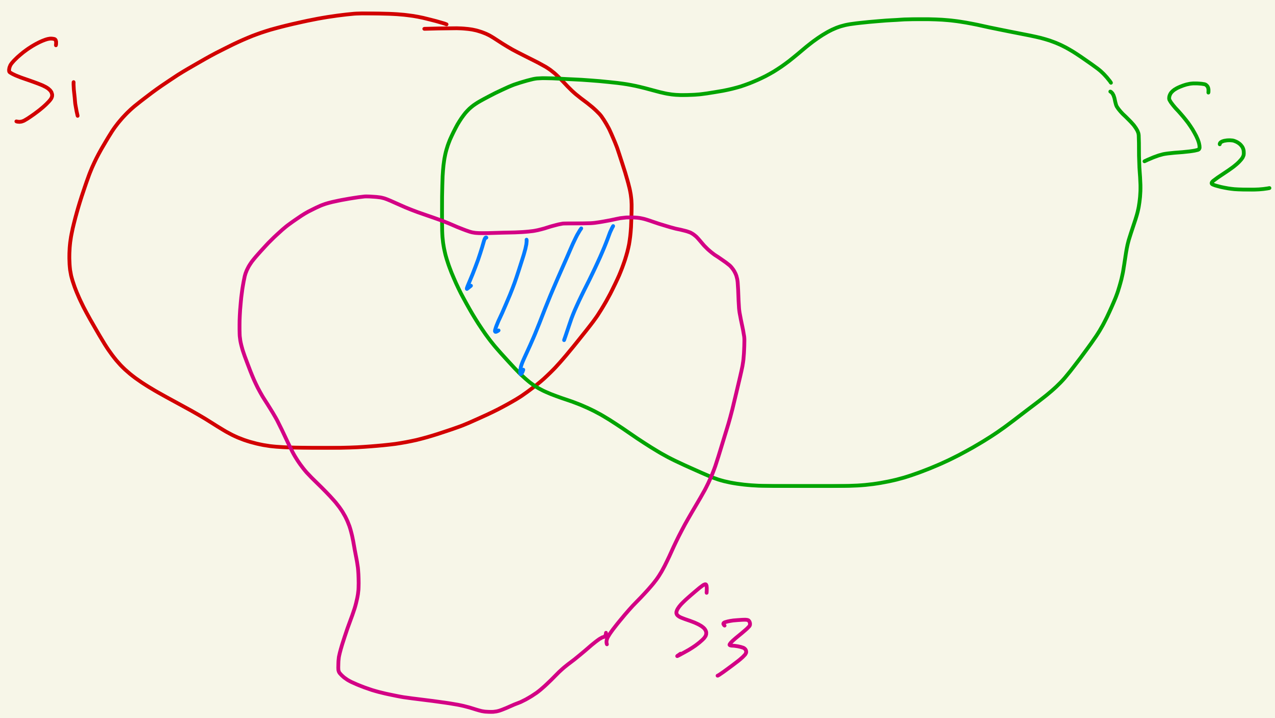


shaded blue is $\bigcup_{S\in\mathcal{A}} S$

Define the <u>intersection over $\mathcal{A}$</u> to be

$$\bigcap_{S \in \mathcal{A}} S = \{ x \mid x \in S \text{ for all } S \in \mathcal{A} \}$$

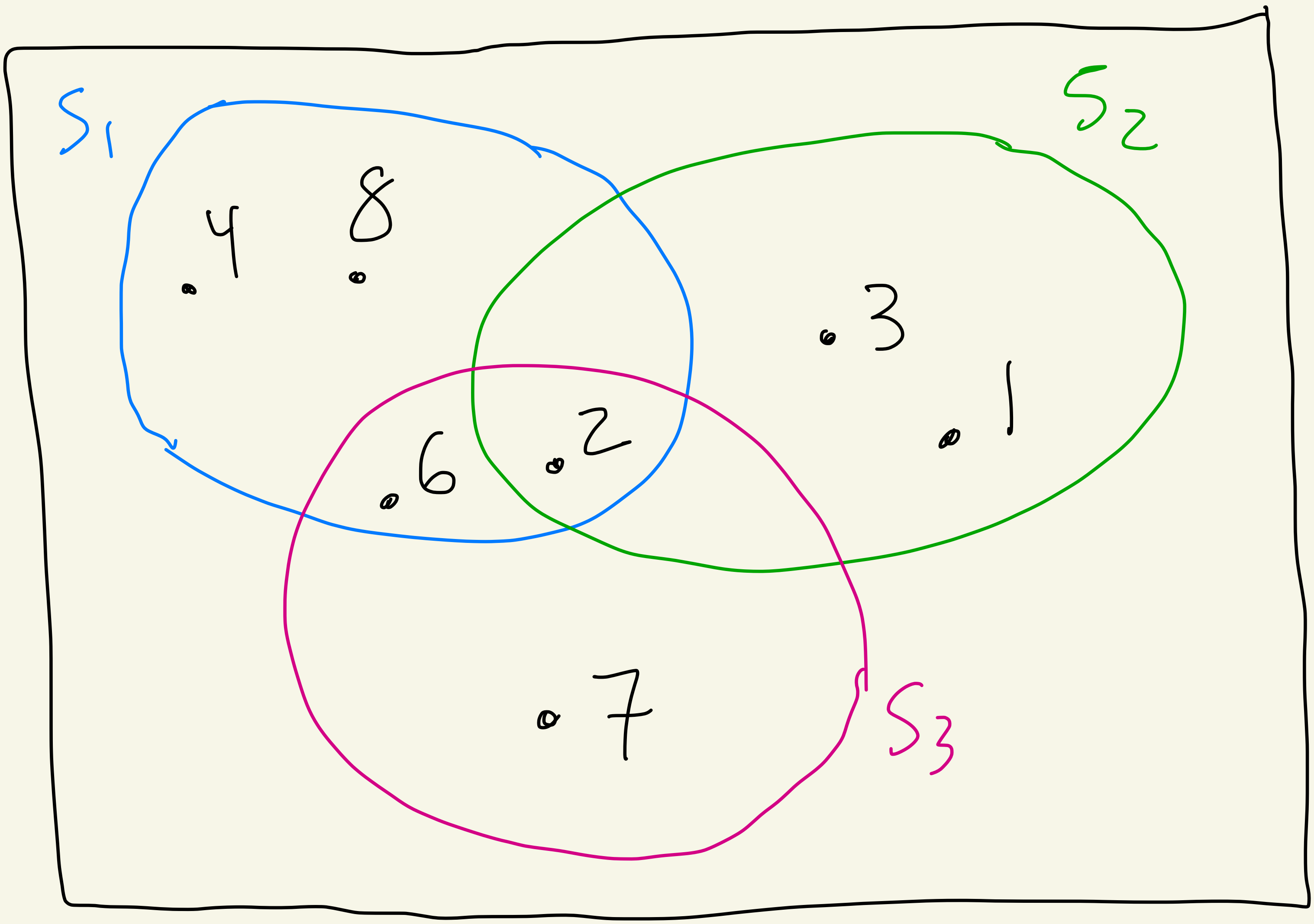
$\boxed{\text{Ex:}}$ $\mathcal{A} = \{ S_1, S_2, S_3 \}$



shaded blue is $\bigcap_{S \in \mathcal{A}} S$

Ex:

$$A = \{ \overbrace{\{2,4,6,8\}}^{S_1}, \overbrace{\{1,3,2\}}^{S_2}, \underbrace{\{2,7,6\}}_{S_3} \}$$



$$\bigcup_{S \in A} S = \{1, 2, 3, 4, 6, 7, 8\}$$

$$\bigcap_{S \in A} S = \{2\}$$

$\boxed{\text{Ex:}}$ $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

$\mathbb{Z}$, $\{\dots, -2, -1, 0, 1, 2, \dots\}$

$$\mathcal{B} = \Big\{ \overbrace{\{n \in \mathbb{Z} \mid |n| \leq k\}}^{S_k} \;\Big|\; k \in \mathbb{N} \Big\}$$

$$= \{ S_k \mid k \in \mathbb{N} \}$$

$$= \{ S_1, S_2, S_3, S_4, \dots \}$$

and $S_k = \{ n \in \mathbb{Z} \mid |n| \leq k \}$

$$S_1 = \{ n \in \mathbb{Z} \mid |n| \leq 1 \} = \{-1, 0, 1\}$$

$$S_2 = \{ n \in \mathbb{Z} \mid |n| \leq 2 \} = \{-2, -1, 0, 1, 2\}$$

$$S_3 = \{-3, -2, -1, 0, 1, 2, 3\}$$

$$S_4 = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$$

And so on...

$$\bigcup_{S \in B} S = \mathbb{Z}$$

$\boxed{\bigcup_{k=1}^{\infty} S_k \text{ is another way to write it}}$

$$\bigcap_{S \in B} S = \{-1, 0, 1\}$$

$\boxed{\bigcap_{k=1}^{\infty} S_k \text{ is another way to write it}}$

Def: Let I be a non-empty set. Suppose for each $\alpha \in I$ there is a corresponding set A_α.

The family

$$\mathcal{A} = \{ A_\alpha \mid \alpha \in I \}$$

is called an indexed family of sets. The set I is called the index set.

If $\alpha \in I$, then α is called the index of A_α.

Ex: Previously we had

$$\mathcal{B} = \{ S_k \mid k \in \mathbb{N} \}$$

Here $\mathcal{B}$ is an indexed family of sets. $\mathbb{N}$ is the index set.

$S_3 \leftarrow$ $\alpha = 3$ is the index of S_3

Def: Let I be a non-empty set. Let $\mathcal{A} = \{ A_\alpha \mid \alpha \in I \}$ be an indexed family of sets.

Define the <u>union over $\mathcal{A}$</u>

$$\bigcup_{\alpha \in I} A_\alpha = \left\{ x \;\middle|\; \begin{array}{l} \text{there exists } \alpha \in I \\ \text{with } x \in A_\alpha \end{array} \right\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \left\{ x \;\middle|\; \begin{array}{l} x \in A_\alpha \text{ for} \\ \text{all } \alpha \in I \end{array} \right\}$$

Ex: In our previous example

$$\mathcal{B} = \{ S_k \mid k \in \mathbb{N} \}$$

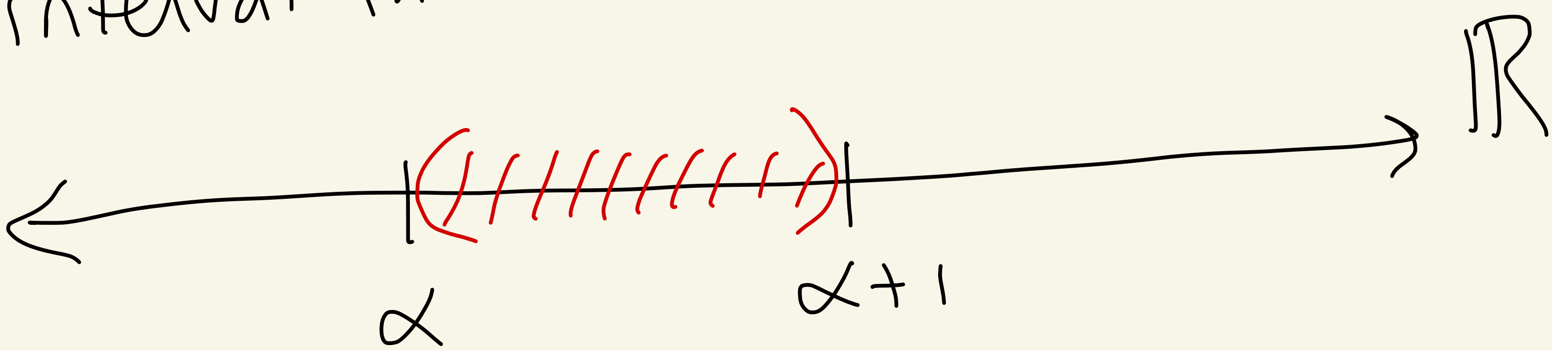
we would write

$$\bigcup_{k \in \mathbb{N}} S_k = \mathbb{Z} \quad \text{and} \quad \bigcap_{k \in \mathbb{N}} S_k = \{-1, 0, 1\}$$

$\boxed{\text{Ex:}}$ Let's make sense of

$$\bigcup_{\alpha \in \mathbb{Z}} (\alpha, \alpha+1) \quad \text{and} \quad \bigcap_{\alpha \in \mathbb{Z}} (\alpha, \alpha+1)$$

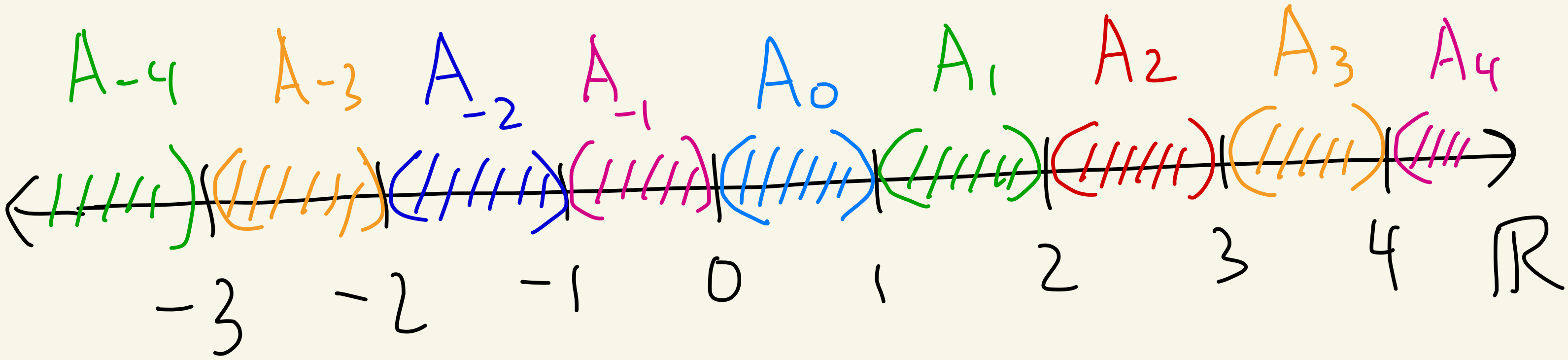
where $(\alpha, \alpha+1)$ means the interval in the real numbers $\mathbb{R}$



$I = \mathbb{Z}$

$A_\alpha = (\alpha, \alpha+1)$

$\mathcal{A} = \{A_\alpha \mid \alpha \in \mathbb{Z}\}$



$$\underbrace{\bigcup_{\alpha \in \mathbb{Z}}}\ A_\alpha = \mathbb{R} - \mathbb{Z}$$

another way to write this is $\bigcup_{\alpha=-\infty}^{\infty} A_\alpha$

its understood α ranges over whole numbers $\mathbb{Z}$

$$\underbrace{\bigcap_{\alpha \in \mathbb{Z}}}\ A_\alpha = \phi$$

another way to write $\bigcap_{\alpha=-\infty}^{\infty} A_\alpha$

its understood α ranges over whole numbers $\mathbb{Z}$

Theorem: Let $\mathcal{A} = \{A_\alpha \mid \alpha \in I\}$ be an indexed family of sets. Let $\alpha_0 \in I$ be a fixed element.

Then:

① $A_{\alpha_0} \subseteq \bigcup_{\alpha \in I} A_\alpha$

② $\bigcap_{\alpha \in I} A_\alpha \subseteq A_{\alpha_0}$

Ex:

$I = \mathbb{N}$

$\alpha_0 = 5$

$A_5 \subseteq \bigcup_{\alpha \in \mathbb{N}} A_\alpha$

$\bigcap_{\alpha \in \mathbb{N}} A_\alpha \subseteq A_5$

proof:

① Pick some $x \in A_{\alpha_0}$.

Then there exists $\alpha \in I$ (namely $\alpha = \alpha_0$) where $x \in A_\alpha$

Thus,

$$x \in \bigcup_{\alpha \in I} A_\alpha = \left\{ y \;\middle|\; \begin{array}{c} \text{there exists} \\ \alpha \in I \text{ with} \\ y \in A_\alpha \end{array} \right\}$$

Therefore,

$$A_{\alpha_0} \subseteq \bigcup_{\alpha \in I} A_\alpha$$

2 You try. $\blacksquare$